Holographic F-theory

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Abstract

The central theme of this thesis is Type IIB supergravity solutions and their field theory duals. The uniqueness of our set up is the inclusion of an AdS factor in the geometry. In traditional F-theory compactifications the non-compact part of spacetime is Minkowski space, by including an AdS factor we may appeal to the AdS/CFT correspondence and probe the dual field theories from the gravity side.

In the first part of this thesis we will be interested in AdS_3 solutions with F-theoretic interpretations. We find the general conditions for the existence of a supersymmetric solution with $(0,2)$ supersymmetry. This is determined by the choice of a 6d Kähler base satisfying a master equation. One may give this equation an F-theoretic interpretation by the inclusion of an auxiliary elliptic fibration which models the varying axio-dilaton as in canonical F-theory compactifications.

The unique family of $(0,4)$ solutions are holographically dual to D3-branes wrapped on curves inside a Calabi–Yau three-fold and correspond to self-dual strings in the 6d $\mathcal{N} = (0,1)$ theory obtained from F-theory on the aforementioned Calabi–Yau threefold. The dual field theory to this set up has been discussed in the literature, but only in the abelian ($N = 1$) case. The power of the AdS/CFT correspondence allows us to make predictions for $N > 1$ which are otherwise inaccessible from the field theory side with current technology. We compute the holographic central charges and show that these agree with the field theory and with the anomalies of self-dual strings in 6d. We complement our analysis with a discussion of the dual M-theory solutions and a comparison of the central charges.

We supplement our $(0,4)$ analysis with a discussion on $(0,2)$ solutions. We discuss three classes of solutions with varying axio-dilaton. It is interesting to note that contrary to the popular F-theory lore, Ricci-flat (i.e. Calabi–Yau) manifolds are not a necessary condition for an F-theory geometry. In each of these classes we compare the holographic central charges with field theory results obtained by using c-extremisation, finding perfect agreement.

In the final chapter of this thesis we complete the classification of AdS_5 solutions in Type IIB by extending the existing classification to allow for vanishing self-dual five-form. AdS_5 solutions with vanishing five-form have been found recently which evaded the previous classification and we show how these solutions fit into the extended classification presented here. We allow throughout for a varying axio-dilaton.

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Chapter 1

Introduction

The two main topics of this thesis are the AdS/CFT correspondence and F-theory. Each of which has been studied in great detail separately. In this thesis we begin the process of uniting these two topics in order to use the power of each simultaneously. For the ease of the reader we start this thesis with an overview of these two main topics. Due to the large amount of research conducted in both of these directions we will only be able to skim the surface of the sea of interesting research in these fields.

1.1 Type IIB supergravity and F-theory

F-theory, first introduced in [2], arises as the non-perturbative completion of Type IIB string theory, as such it is prudent to first discuss Type IIB string theory, in particular its low energy limit Type IIB supergravity. There are many nice reviews of F-theory, a non-exhaustive list is; [3–5].

1.1.1 Type IIB supergravity

Type IIB string theory is a theory of supersymmetric closed oriented strings in 10d. It possesses $\mathcal{N} = (2, 0)$ supersymmetry, with two Majorana-Weyl supercharges with the same (positive) chirality. The associated spinors parametrising the supersymmetry variations are also Majorana-Weyl with negative chirality. The supersymmetry algebra admits a $U(1)_R$ R-symmetry rotating the two supercharges.

The bosonic massless sector of the theory contains: the Neveu-Schwarz-Neveu-Schwarz (NSNS) sector containing a metric g, a scalar field called the dilaton Φ and a two-form B; the Ramond-Ramond (RR) sector containing a scalar known as the axion $C^{(0)}$, a two form potential $C^{(2)}$ and a four form potential $C^{(4)}$ with self-dual field strength. The fermionic sector contains two gravitini, Ψ and a dilatini, λ. Due to the self-duality of the field strength of the four-form potential there is no canonical covariant Lagrangian formulation of the theory.

The two scalars of the theory parametrise a scalar manifold, $\mathcal{M}$ which must have holonomy containing the $U(1)_R$ automorphism group, see for example [6]. Moreover the scalar manifold must be negatively curved and locally isometric to a symmetric manifold, this leaves a single choice for a simply connected manifold, namely the upper-half plane

$$\mathcal{M} = SL(2,\mathbb{R})/U(1)_R \tag{1.1}$$

equipped with the $SL(2,\mathbb{R})$ invariant metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}\tau \mathrm{d}\bar{\tau}}{(\mathrm{Im}\,[\tau])^2} \ . \tag{1.2}$$

In fact by quantum corrections this symmetry is reduced to $SL(2,\mathbb{Z})$ in the full non-perturbative quantum theory[1]. The scalars are combined into the complex scalar, referred to throughout as the axio-dilaton,

$$\tau = C^{(0)} + \mathrm{i}e^{-\Phi} = \tau_1 + \mathrm{i}\tau_2 \ , \tag{1.3}$$

which transforms under the global $SL(2,\mathbb{R})$ via Möbius transformations,

$$\tau \to \frac{a\tau + b}{c\tau + d} \ , \qquad ad - bc = 1 \ . \tag{1.4}$$

The remaining non-scalar bosonic fields must then organise themselves into linear representations of $SL(2,\mathbb{R})$. The metric in Einstein frame and the five-form are invariant whilst the two two-forms transform as a doublet;

$$\begin{pmatrix} C^{(2)} \\ B \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C^{(2)} \\ B \end{pmatrix} \ . \tag{1.5}$$

It is convenient to combine the two three-form field strengths into one complex one as

$$G = \frac{\mathrm{i}}{\sqrt{\tau_2}}(\tau \mathrm{d}B - \mathrm{d}C^{(2)}) \ , \tag{1.6}$$

and the scalars into a complex one-form field strength

$$P = \frac{\mathrm{i}}{2\tau_2}\mathrm{d}\tau \ . \tag{1.7}$$

The covariant derivative is gauged with respect to the local $U(1)_R$ symmetry

$$\mathcal{D} = \nabla - \mathrm{i}qQ \ , \quad Q = -\frac{1}{2\tau_2}\mathrm{d}\tau_1 \ . \tag{1.8}$$

[1] It is easy to see that $SL(2,\mathbb{R})$ cannot be the symmetry group of the full quantum theory. Tree level string perturbation theory amplitudes depend on the coupling constant, but by an $SL(2,\mathbb{R})$ transformation we may set the dilaton, and hence the string coupling to any value. Clearly this is incompatible.

with the charges of all the fields given in table 1.1.1, The connection defines a line

Field	$U(1)_R$ charge, q
F_5	0
g	0
G	1
P	2
ϵ	$\frac{1}{2}$
Ψ	$\frac{1}{2}$
λ	$\frac{3}{2}$

Table 1.1: $U(1)_R$ charges of the fields of Type IIB supergravity

bundle, denoted $\mathcal{L}_D$ in the literature called the duality bundle. This encodes the varying axio-dilaton profile in F-theory and it is clear it is trivial when τ is constant.

Without the existence of a covariant Lagrangian we present the covariant equations of motion[2]. The bosonic equations of motion consist of the Einstein equation

$$R_{MN} = 2P_{(M}P^*_{N)} + \frac{1}{96}F_{5MP_1..P_4}F_{5N}{}^{P_1..P_4} + \frac{1}{8}\left(2G_{(M}{}^{P_1P_2}G^*_{N)P_1P_2} - \frac{1}{6}g_{MN}G^{P^1..P_3}G^*_{P_1..P_3}\right) \quad (1.9)$$

and the flux equations of motion and Bianchi identities

$$\begin{aligned} \mathcal{D} * G = P \wedge *G^* + \mathrm{i}F \wedge G \ , \quad \mathcal{D} * P = -\frac{1}{4}G \wedge *G \ , \quad F_5 = *F_5 \, , \\ \mathcal{D}P = 0 \, , \quad \mathcal{D}G = -P \wedge G^* \, , \quad \mathrm{d}F_5 = \frac{\mathrm{i}}{2}G \wedge G^* \, . \end{aligned} \quad (1.10)$$

Finally the fermionic supersymmetry equations are

$$\begin{aligned} \delta\psi_M &= \mathcal{D}_M\epsilon + \frac{\mathrm{i}}{192}\Gamma^{P_1...P_4}F_{MP_1...P_4}\epsilon \\ & -\frac{1}{96}\left(\Gamma_M{}^{P_1..P_3}G_{P_1..P_3} - 9\Gamma^{P_1P_2}G_{MP_1P_2}\right)\epsilon^c \ , \quad (1.11) \\ \delta\lambda &= \mathrm{i}\Gamma^M P_M \epsilon^c + \frac{\mathrm{i}}{24}\Gamma^{P_1..P_3}G_{P_1...P_3}\epsilon \ . \quad (1.12) \end{aligned}$$

1.1.2 F-theory

Consider a p-brane, the classical brane solution is

$$\mathrm{d}s^2_E = H(r)^{\frac{p-7}{8}}\mathrm{d}x^2_{||} + H(r)^{\frac{p+1}{8}}\mathrm{d}x^2_{\perp} \ , \quad H(r) = 1 + \frac{\alpha}{r^{7-p}} \ , \mathrm{e}^{\Phi(r)} = H(r)^{\frac{3-p}{4}} \quad (1.13)$$

where $H(r)$ is a harmonic function on $\mathrm{d}x^2_\perp$. Clearly for $p = 7$, and D7-branes something is amiss and the above solution is not admissible. The harmonic function

[2] Our conventions will be those of [7]

is logarithmic in this case. Close to the brane it takes the form

$$\tau(z) = \frac{1}{2\pi \mathrm{i}} \log z + ... \tag{1.14}$$

which is singular at $z = 0$ and undergoes monodromy $\tau \to \tau + 1$ around this locus. The monodromy seems to preclude a consistent interpretation of the background, however the $SL(2,\mathbb{Z})$ symmetry of the theory comes to the rescue. Upon encircling a D7-brane the full background transforms by an $SL(2,\mathbb{Z})$ transformation; the monodromy is simply a symmetry of the theory. Upon accepting the necessity of including $SL(2,\mathbb{Z})$ transformations we are forced to accept also the existence of more general 7-branes. There must exist $[p,q]$ 7-branes and the corresponding (p,q)-strings[3] ending on these $[p,q]$ 7-branes.

A general $[p,q]$ 7-brane is then manifestly non-perturbative as the dilaton becomes large near to the brane. Of course one may use the $SL(2,\mathbb{Z})$ symmetry to map a $[p,q]$ 7-brane to a D7-brane and the two geometries are indistinguishable. However for mutually non-local $[p,q]$ 7-branes there is no duality frame in which all the branes can be simultaneously transformed into D7's. In fact for a consistent compact geometry vanishing of the total global brane charge (tadpole condition) implies that mutually non-local $[p,q]$ 7-branes are necessary. This is where the difficulty in F-theory arises. If one wants to include perturbative D7-branes, one is forced to include also the non-perturbative $[p,q]$ 7-branes. In certain regions (near to D7-branes) the theory is weakly coupled but in other regions it is strongly coupled.

The vital understanding of this problem was provided in [2]. One identifies the $SL(2,\mathbb{Z})$ symmetry of Type IIB with the geometric $SL(2,\mathbb{Z})$ action on the complex structure of a two-torus, whilst the axio-dilaton τ is the complex structure of this fictitious elliptic curve. The F-theory conjecture is; the physics of Type IIB with 7-branes on an n-fold B_n is encoded in the geometry of the $n+1$-fold Y_{n+1} which is an elliptic fibration over B_n, $\mathbb{E}_\tau \hookrightarrow Y_{n+1} \to B_n$. The elliptic fiber is not part of spacetime, it is a bookkeeping device, see figure 1.1.2 for a pictorial representation. At the location of 7-branes τ diverges which corresponds to the shrinking of a cycle. A (p,q) cycle shrinking to zero volume sources a $[p,q]$ 7-brane in the geometry. F-theory is then equivalent to the study of elliptic fibrations.

Before proceeding a few comments are in order. Though it seems that the more fundamental theory should be a 12d theory containing an elliptically fibered manifold, on which one may compactify to obtain Type IIB, this is not the case. Firstly there is no 12d supergravity with signature $(1,11)$ preserving 32 supercharges, moreover the volume modulus of the T^2 has no analogue in Type IIB. Instead an alter-

[3]A (p,q) string carries p units of electric B-charge and q units of electric $C^{(2)}$ charge with a $(1,0)$ string the perturbative $F1$-string.

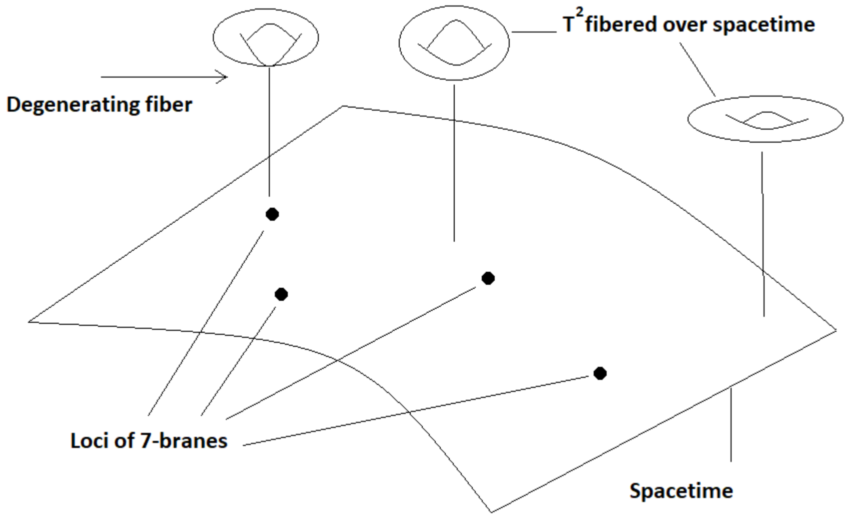

Figure 1.1: The torus is fibered over the ten-dimensional spacetime. The presence of 7-branes is determined by the degeneration locus of the elliptic curve. In general the locus may be a surface rather than a set of points as shown here.

native definition is via M/F-duality.

$$\text{F-theory on } Y \times S^1 \;\equiv\; \text{M-theory on } Y \ .$$

One reduces the M-theory solution along a cycle of the elliptic fibration and then T-dualizes along the remaining cycle to obtain a Type IIB solution with axio-dilaton given by the complex structure of the elliptic fibration.

We saw that all the F-theory data is encoded in the elliptic fibration. The natural language in which to discuss elliptic fibrations is algebraic geometry. Consider Y_{n+1} as from before and assume that it admits a section, $s : B_n \to \mathbb{E}_\tau$. If the elliptic fibration has no singular fibers the fibration is in fact topologically trivial and the axio-dilaton is constant everywhere. Here we will be interested in non-trivial fibrations, which necessarily include so called Kodaira singular fibers [8, 9][4]. One can model an elliptic fibration as a Weierstrass model. The elliptic fibration is modeled by the hypersurface

$$y^2 = x^3 + fxw^4 + gw^6 \tag{1.15}$$

where f, g are sections of $K_{B_n}^{-4}$ and $K_{B_n}^{-6}$ respectively, with K_{B_n} the canonical class of the base. The coordinates $[w, x, y]$ satisfy the standard projective relations in $\mathbb{P}^{1,2,3}$. The class of the section of the elliptic fibration will be denoted by σ. For a more in depth review of elliptic fibrations and their geometry and more specifically the intersection theory used in the following, we refer the reader to *e.g.* [3, 10, 11]. We have provided some additional details on elliptic fibrations of Calabi–Yau three-

[4] As a cautious remark, despite the name, these are the resolved fibers above singular loci of the fibration.

folds in appendix B.2.1. For each point in the base, this equation defines an elliptic curve, whose complex structure can be determined via the j-function, which in turn depends on f and g. Singularities in the elliptic fibration are characterised by the vanishing of the discriminant Δ of the Weierstrass equation,

$$\Delta = 4f^3 + 27g^2 = 0\,, \tag{1.16}$$

which defines complex codimension one loci in B_n. The type of singular fibers that can occur were classified by Kodaira–Néron, and are characterised in terms of the order of vanishing of (f, g, Δ) along the discriminant locus. The simplest Kodaira fiber is I_1, which has

$$I_1: \qquad \mathrm{ord}(f, g, \Delta) = (0, 0, 1)\,. \tag{1.17}$$

The I_1 singular fiber corresponds to a single D7-brane. The worldvolume of the 7-branes is $\Delta \times \mathbb{M}^{1,d}$. For a given Weierstrass model, the complex structure τ of the elliptic curve can be extracted from the Jacobi j-function

$$j(\tau) = -1728 \frac{(4f)^3}{\Delta}\,, \tag{1.18}$$

by expanding as

$$j(\tau) = \frac{1}{q} + 744 + \cdots\,, \tag{1.19}$$

where $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$. Using the asymptotic expansion along the loci where $\Delta = 0$ one can extract the local behaviour of τ. For example, the axio-dilaton close to a 7-brane wrapping the local divisor $z = 0$ in the base B has the profile

$$\tau = \frac{1}{2\pi \mathrm{i}} \log z + \cdots\,, \tag{1.20}$$

which agrees with the form obtained from considering the supergravity brane solution (1.14). Specifically this singular behaviour of the axio-dilaton implies that the metric on the base will have singularities.

In the present context we are interested in solutions to the effective theory. Naively we would define the F-theory supergravity solutions on Y in terms of Type IIB supergravity, on B_n including τ which varies over B_n. However when the elliptic fibration has singularities as in (1.20), the metric that is induced on the base B_n is expected to be singular. In the case of K3 surfaces, this can be made explicit for non-compact [12] and for compact K3s [13, 14], who also give a precise measure for the divergence of the curvature scalar close to the singular fibers. Thus a supergravity approach seems at first sight to be somewhat questionable.

1.1.3 Field theories with varying coupling

The main motivation for wishing to understand F-theoretic supergravity solutions is to better understand field theories with varying couplings. The addition of a varying coupling constant requires a variant of the topological twist known as the topological duality twist. This was introduced in [15] for the abelian theory and generalised to non-abelian theories in [16] via M-theory.

Let us consider the case of abelian $\mathcal{N} = 4$ SYM on a curve, C with varying coupling constant, as this is the best understood. The constant coupling analogue of this reduction was performed in [17]. By including a varying coupling the twist must include both the usual topological twist of the structure group of the curve, an R-symmetry factor and the additional *duality* or *bonus symmetry* of the abelian theory. The variation of the coupling τ may be understood in terms of a non-trivial line bundle which gives the bonus symmetry. We denote this line bundle by $\mathcal{L}_D$ and is defined over the section of spacetime over which τ varies. Under an $SL(2,\mathbb{Z})$ transformation the coupling transforms under a Möbius transformation as in (1.4). A field of $U(1)_D$ charge q transforms as

$$\phi \to \mathrm{e}^{\mathrm{i}q\alpha(\tau)}\phi \ , \qquad \mathrm{e}^{\mathrm{i}\alpha(\tau)} = \frac{c\tau + d}{|c\tau + d|} \ . \tag{1.21}$$

That is it transforms as a section of the q-th power of the $\mathcal{L}_D$. The one form connection is given by Q in (1.8).

The 4d spacetime is taken to be $\mathbb{R}^{1,1} \times C$ which breaks the $SO(1,3)$ Lorentz symmetry to $SO(1,1) \times U(1)_C$. The topological twist requires one to turn on a $U(1)$ R-symmetry background gauge field with which one twists the theory. Consider the decomposition of the $SU(4)_R$ R-symmetry of $\mathcal{N} = 4$ SYM as

$$SU(4)_R \to SO(4)_T \times U(1)_R \ . \tag{1.22}$$

Performing a conventional topological twist, twists the $U(1)_R$ with the $U(1)_C$. In the varying coupling case the supercharges transform non-trivially under $U(1)_D$ and therefore to have well-defined supercharges in the dimensionally reduced theory one must also twist with $U(1)_D$. This is precisely what is meant when we refer to the topological duality twist. For a D3-brane on C inside a Calabi–Yau three-fold the twist preserving $(0,4)$ is

$$T_C^{twist} = \frac{1}{2}(T_C + T_R) \ , \quad T_D^{twist} = \frac{1}{2}(T_D + T_R) \ . \tag{1.23}$$

1.1.4 D3-branes in F-theory and 2d $(0,4)$ SCFTs

Having briefly discussed the necessity of performing the topological duality twist one can reduce the 4d fields to 2d with the above twisting, and compute the spectrum of the 2d $\mathcal{N} = (0,4)$ SCFT. The field content for these 2d SCFTs was worked out in [18], where in particular the abelian zero mode spectrum and the left and right central charges were computed.

The zero mode spectrum in terms of $(0,4)$ multiplets was found to be

$$\begin{array}{c|c|c} (0,4) \text{ multiplet} & \text{Multiplicity} & (c_R, c_L) \\ \hline \text{Hyper} & \frac{1}{2}C \cdot C + \frac{1}{2}c_1(B) \cdot C & (6,4) \\ \text{Twisted Hyper} & 1 & (6,4) \\ \text{Fermi} & \frac{1}{2}C \cdot C - \frac{1}{2}c_1(B) \cdot C + 1 & (0,2) \end{array} \,. \tag{1.24}$$

In addition, one has half-Fermi multiplets arising from 3-7 strings, which contribute $c_{37} = 8c_1(B) \cdot C$ to the left-moving central charge. The left and right central charges are computed by summing the contributions from each multiplet with their appropriate multiplicity, and are given by [18, 19]

$$c_R = 3C \cdot C + 3c_1(B) \cdot C + 6\,, \tag{1.25}$$

$$c_L = 3C \cdot C + 9c_1(B) \cdot C + 6\,. \tag{1.26}$$

Notice that upon using the adjunction formula (3.37), the right central charge may be rewritten as

$$c_R = 6\,(g + c_1(B) \cdot C)\ , \tag{1.27}$$

which is manifestly a multiple of 6, as expected generically for $(0,4)$ SCFTs with small superconformal algebra [20]. Under M/F-duality, this is equivalent to M5-branes wrapped on the elliptic surface $\widehat{C} = \pi^*(C)$ in the Calabi–Yau threefold. The 2d spectrum obtained from a single M5-brane wrapped on an elliptic surface was also determined in [18] as

$$\begin{array}{c|c|c} (0,4) \text{ multiplet} & \text{Multiplicity} & (c_R, c_L) \\ \hline \text{Hyper} & \frac{1}{2}C \cdot C + \frac{1}{2}c_1(B) \cdot C + 1 & (6,4) \\ \text{Fermi} & \frac{1}{2}C \cdot C - \frac{1}{2}c_1(B) \cdot C + 1 & (0,2) \\ \text{Half-Fermi} & 8c_1(B) \cdot C & (0,1) \end{array} \,. \tag{1.28}$$

Here, the half-Fermi multiplets arise directly from the reduction of the 6d $\mathcal{N} = (2,0)$ tensor multiplet. This spectrum matches that of the D3-brane wrapped on C and therefore the left and right central charges are also given by (1.25). We remind the reader that these central charges are computed for a single D3-brane, *i.e.* $N = 1$. In the following we will compute these central charges holographically for general N.

1.1.5 c-extremization

In the previous section we have given the left and right moving central charges of a particular class of 2d $\mathcal{N} = (0,4)$ SCFTs as computed in [18]. The large amount of supersymmetry lead to a simplification in the computation; the UV R-symmetry descended to the IR R-symmetry without any mixing. With less supersymmetry, in particular $\mathcal{N} = (0,2)$ that we shall study later, things become more difficult. Along the RG flow the UV R-symmetry may mix with the other global $U(1)$ isometries in the theory whence in the IR it is some complicated linear combination of them all. This phenomenon is familiar in the case of 4d SCFTs and was resolved in [21] by the process of a-maximization. Such an extremization principle also exists in 2d and goes by the name of c-extremization, [17,22], and we shall review it in the following section.

We begin with a small recap about anomalies in 2d. Gauge theories in 2d may have both gauge and gravitational anomalies but are forbidden to have mixed gauge-gravitational anomalies [17]. Consider a theory with continuous global symmetry G whose abelian part is $U(1)^M$. The theory necessarily has conserved current operators $J^I, I = 1, .., M$ and a conserved stress-energy tensor. Upon coupling the theory to a curved background, $g_{\mu\nu}$ and background vector fields A^I_μ, the anomalous variations of current conservation are

$$\nabla_\mu T^\mu{}_\nu = \frac{k}{96\pi}\epsilon^{\mu\rho}\partial_\mu\partial_\sigma\Gamma^\sigma_{\nu\rho}\,, \qquad \nabla^\mu J^I_\mu = \sum_J \frac{k^{IJ}}{8\pi}F^M_{\mu\nu}\epsilon^{\mu\nu}\,, \tag{1.29}$$

with F^I the field strength of the background vector field A^I and $\Gamma^\sigma_{\mu\rho}$ the Levi-Civita connection for the background metric $g_{\mu\nu}$. The anomalous variations are encoded in the constant coefficients k^{IJ}, and k known as ‘t Hooft anomaly coefficients.

For a weakly coupled theory the ‘t Hooft anomaly coefficients only receive contributions from chiral fermions and bosons and may be computed exactly by one-loop diagrams with two current insertions. Regardless of a weakly coupled Lagrangian description, and assuming the symmetries are not broken along the RG-flow, the anomaly coefficients are well-defined by the anomalous conservation equations (1.29). Under these assumptions, ‘t Hooft aomaly matching implies that one may calculate these anomalies in the strongly coupled IR by using only the weakly coupled UV description.

For a conformal theory the anomaly coefficients are related to central terms in the conformal and current algebras in flat space. In particular the algebra implies

$$c_R = 3k^{RR} \tag{1.30}$$

where c_R is the right-moving central charge of the theory. This therefore relates the central charge with the exact R-symmetry and its ‘t Hooft anomaly coefficient.

This makes manifest the importance of knowing the exact R-symmetry in the IR; we may determine the central charge from it. The idea is to characterize the exact R-symmetry in terms of anomalies which are invariant under the RG flow and thus independent of a detailed knowledge of the IR fixed point. Consider a trial R-current

$$R_{\text{trial}} = R_0 + \sum_I \epsilon_I J^I \tag{1.31}$$

where R_0 is an arbitrary choice of R-symmetry and the J^I are all the other abelian currents in the theory. We may then construct a trial central charge as

$$c_R^{\text{trial}}(\epsilon) = 3k_{RR}^{\text{trial}} = \sum_{\text{fields}} q_R^2 \ , \tag{1.32}$$

with q_R the R-charge of the field under the trial R-symmetry. At an IR $\mathcal{N} = (0,2)$ fixed point there are no mixed gauge anomalies between the superconformal R-current and other (flavour, Baryonic, etc.) global abelian symmetries,

$$k_{RI} = 0 \ , \quad \forall I \neq R \ . \tag{1.33}$$

It is easy to see that this condition is equivalent to c being extremised with respect to the mixing parameters ϵ_I in (1.31)

$$\frac{\partial c_R^{\text{trial}}(\epsilon^*)}{\partial \epsilon_I} = 0 \ . \tag{1.34}$$

As c_R^{trial} is a quadratic function there is a unique solution. We conclude that the exact superconformal R-symmetry of the IR fixed point is the one that extremizes (1.32) thereby also giving the central charge.

As presented above, the formulas are adapted for flowing from a 2d UV fixed point to a 2d IR fixed point. During this thesis we will be interested in compactifying a 4d SCFT at a UV fixed point on a Riemann surface, Σ_g which then flows to a 2d IR fixed point. For clarity later, we shall present a modified trial central charge to account for this difference. One performs a topological twist by turning a background gauge field along the generator

$$T_{\text{twist}} = \frac{\kappa}{2} + \sum_I t_I T_I \tag{1.35}$$

with T_R a representative R-symmetry and T_I all other global abelian currents in the theory. It is necessary to quantise the flux through the background field which we are turning on, viz.

$$F = T_{\text{twist}} \text{dvol}(\Sigma_g) \ , \quad \frac{1}{2\pi}\int_{\Sigma_g} F \cdot \mathcal{O} = \eta(\Sigma_g) T \cdot \mathcal{O} \equiv n\mathcal{O} \tag{1.36}$$

where $\mathcal{O}$ is any gauge invariant operator and

$$\eta(\Sigma_g) = \begin{cases} 2|g-1| \ , & g \neq 1 \\ 1 \ , & g = 1 \end{cases} \tag{1.37}$$

The standard Dirac quantisation imposes that the constant n appearing in (1.36) is integer. The trial central charge is

$$c_R^{\text{trial}} = -3\eta(\Sigma_g) \sum_{\text{fields}} q_{\text{twist}} q_R^2 \tag{1.38}$$

where as before q_R is the R-charge of the field and q_{twist} is the charge under the topological twist. This formula relies on the index theorem result

$$n_r - n_l = -q_{\text{twist}} \eta(\Sigma_g) \tag{1.39}$$

which gives the difference between the number of right-moving and left-moving 2d chiral massless fermions.

1.2 Holography

The holographic principle states that the entire information content of a quantum gravity theory in a given volume can be encoded in an effective theory at the boundary surface of this volume. The two theories though physically different are equivalent. One can perform a measurement in one theory and there is an equivalent measurement one can perform in the second dual theory.

The AdS/CFT correspondence is a particular example of the holographic principle. It was first conjectured in [23] where Type IIB string theory on $\text{AdS}_5 \times S^5$ was conjectured to be dual to 4d $\mathcal{N} = 4$ Super Yang–Mills (SYM)[5]. The motivation for such a duality comes from two complementary views of branes.

Consider a stack of N coincident D3-branes. From the point of view of the string field theory on the branes there are two possible excitations; open and closed strings. Open strings end on the D3-branes and are excitations of the branes whilst closed strings are excitations of the bulk spacetime. In the low energy limit the interactions between the strings are suppressed and one obtains two theories, the bulk theory and the theory on the brane. The effective field theory on the brane is 4d $\mathcal{N} = 4$ $SU(N)$ SYM, whilst the bulk theory is free Type IIB supergravity. On the other hand one can consider the same setup from the supergravity point of view. The low energy theory is obtained by taking the near horizon limit, that is the limit in which the radial distance to the stack of branes goes to zero $r \to 0$. In this limit one is left

[5] In fact more dualities were conjectured there, but $\text{AdS}_5 \times S^5$ is the most detailed and so we focus on this in this section.

with free Type IIB supergravity in the bulk and supergravity on $\text{AdS}_5 \times S^5$. The two viewpoints are identical and therefore one concludes

$$4d\ SU(4)\ \mathcal{N} = 4\ \text{SYM} \quad \Leftrightarrow \quad \text{Type IIB on AdS}_5 \times S^5 \tag{1.40}$$

Notice the agreement of the symmetries on the two sides. The isometry group of AdS_5 is $SO(2,4)$ which is also the conformal group in 4d. Moreover the five-sphere has isometry group $SO(6) \simeq SU(4)$ which is the R-symmetry of the $\mathcal{N} =$ SYM. One can perform more checks of the duality in this case, for example one may compute correlation functions of the two theories and see that they agree, see [24,25] for early clarifications of the duality.

To fully appreciate the correspondence one should look at the relation between the partition functions of the two theories. The string theory background is the product of a compact manifold (S^5 in this example) and a manifold, X_{d+1} with boundary (AdS_5 in this case). It is on the boundary, ∂X_{d+1}, that the field theory is defined. The holographic dictionary states that for every field ϕ in the supergravity theory there is an associated operator $\mathcal{O}$ of the conformal field theory. In order to evaluate the partition function for this string theory one must give boundary conditions for the fields ϕ. Let $\phi^{(0)}$ be the boundary value of ϕ on ∂X_{d+1}. The partition function is then

$$Z_{string}[\phi^{(0)}] = \int_{\phi|_{\partial X_d}\phi=\phi^{(0)}} D\phi \mathrm{e}^{-S[\phi]} \tag{1.41}$$

with $S[\phi]$ the action functional for the string theory and the subscript on the integrations imposes the relevant boundary conditions. The correspondence conjectures that the above partition function equals the generating functional of correlation functions in the conformal field theory,

$$Z_{CFT}[\phi^{(0)}] = \left\langle \exp\left\{ \int_{\partial X_{d+1}} \text{dvol}(\partial X_{d+1}) \mathcal{O}\phi^{(0)} \right\} \right\rangle \ . \tag{1.42}$$

The boundary values of the supergravity fields act a sources for the operators of the conformal field theory.

This is conjectured to hold for any number, N of coincident branes, but in practice one is unable to compute the partition function for arbitrary N due to stringy corrections. In the large N limit one can trust the supergravity approximation. The partition function reduced to the sum of the exponential of the supergravity action functional evaluated on all field configurations satisfying the supergravity equations of motion subject to the boundary condition, that is

$$Z_{string}^{\text{tree-level}}[\phi^{(0)}] = \sum \exp\left(-S[\phi^{cl}(\phi^{(0)})]\right) \ . \tag{1.43}$$

1.2.1 Weyl Anomaly

In a conformal field theory a operator of particular importance is the stress-energy tensor $T_{\mu\nu}$[6]. The corresponding bulk field is the boundary metric, $g^{(0)}$. The metric on X_{d+1}, G, does not however uniquely fix the boundary metric, instead it determines a conformal equivalence class of metrics on the boundary,

$$g_{(0)} \sim \mathrm{e}^{2\sigma(x)} g_{(0)} \tag{1.44}$$

with $\sigma(x)$ an arbitrary function. The boundary metric is the residue of a second order pole of the bulk metric. To obtain such a representative one should pick a function r on X_{d+1} with a simple zero on the boundary. Then $r^2 G$ restricted to the boundary gives a finite metric on the boundary. Different choices of the function r determines the different conformally equivalent metrics.

Naively the trace of the trace of the stress-energy tensor should decouple, as for a (classically) conformally invariant theory it vanishes. Instead we find that in order to regularise the partition function so as to obtain a finite effective action one must pick an arbitrary representative of the conformal class of metrics. Conformal invariance is thus explicitly broken by a *Weyl anomaly*. This Weyl anomaly is present in field theories in even dimension. Later in this thesis we will use this anomaly as evidence for new duals pairs that we propose in this thesis. As we shall use the results as evidence for the matching of dual pairs we shall give an overview of the derivation of the Weyl anomaly in the remainder of this section.

As the correlation functions we must compute only depend on the stress-energy tensor the relevant part of the bulk action is the gravitational part and all other fields may be set to zero. One needs to only study the classical supergravity equations of motion, and therefore for the theories under consideration this is just the Einstein equation with cosmological constant and the Gibbons-Hawking-York term. To this end pick a metric $g^{(0)}$ on ∂X_{d+1} in the given equivalence class. One can put the metric in Fefferman-Graham form [26],

$$G_{MN}\mathrm{d}x^M \mathrm{d}x^N = \frac{l^2}{4r^2}\mathrm{d}r^2 + \frac{1}{r} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{1.45}$$

where the tensor g has limit $g^{(0)}$ as one approaches the boundary at $r = 0$. The length l is determined by the cosmological constant. One may then solve Einstein's equations in a series expansion. In odd dimension one has

$$g = g^{(0)} + r g^{(2)} + ... \tag{1.46}$$

where $g^{(k)}$ is given in terms of the boundary metric $g^{(0)}$ (and its derivatives) by the

[6] We use indices $M = (r, \mu)$ for spacetime.

Einstein equation. In even dimensions this procedure breaks down at order $d/2$ in r whereupon a logarithmic term appears,

$$g = g^{(0)} + rg^{(2)} + .. + r^{d/2}g^{(d)} + r^{d/2}\log r h^{(d)} + ... \; . \tag{1.47}$$

The terms $g^{(2)}, ..g^{(d-2)}, h^{(d)}$ are uniquely determined in terms of $g^{(0)}$ and its derivatives, whilst only the trace and covariant derivative of $g^{(d)}$ is determined. Explicit expressions for the coefficients may be found in appendix A of [27].

One should now compute the on-shell action. The Einstein action with boundary term is

$$S[G] = \frac{1}{16\pi G_N^{(d+1)}}\left(\int_{X_{d+1}} \mathrm{dvol}(X_{d+1})(R+2\Lambda) + \int_{\partial X_{d+1}} \mathrm{dvol}(\partial X_{d+1}) 2\nabla_\mu n^\mu\right) \; . \tag{1.48}$$

Of course due to the infinite volume of X_{d+1} one must regularise the integral. To do so one restricts the bulk integral to the domain $r > \epsilon$ and evaluates the boundary integral at $r = \epsilon$ with ϵ a small positive cutoff. The regularised integral on the boundary becomes

$$S[G]_{\mathrm{reg}} = \frac{1}{16\pi G_N^{(d+1)}}\int \mathrm{d}^d x \mathcal{L} \tag{1.49}$$

with

$$\mathcal{L} = \frac{d}{l}\int_\epsilon^\infty \mathrm{d}r r^{-d/2-1}\sqrt{\det g} + r^{-d/2}\left(-\frac{2d}{l}\sqrt{\det g} + \frac{4}{l} r\partial_r\sqrt{\det g}\right)\Big|_{r=\epsilon} \tag{1.50}$$

where the first contribution is from the bulk integral and the remainder is from the Gibbons-Hawking-York term. For d odd it follows from the above discussion that $\sqrt{\det g}$ admits a power series expansion with covariant coefficients, whilst for d even this is only true up to an including the $r^{d/2}$ terms. In the two cases one may write the Lagrangian as

$$\mathcal{L} = \sqrt{\det g^{(0)}}\left(a^{(0)}\epsilon^{-d/2} + a^{(2)}\epsilon^{-d/2+1} + ... + a^{(d-1)}\epsilon^{-1/2}\right) + \mathcal{L}_{finite} \tag{1.51}$$
$$\mathcal{L} = \sqrt{\det g^{(0)}}\left(a^{(0)}\epsilon^{-d/2} + a^{(2)}\epsilon^{-d/2+1} + ... + a^{(d-2)}\epsilon^{-1} - a^{(d)}\log\epsilon\right) + \mathcal{L}_{finite} \tag{1.52}$$

for d odd and even respectively. In the $\epsilon \to 0$ limit the terms $\mathcal{L}_{finite}$ are finite. All the $a^{(k)}$ terms are covariant and so may be cancelled by the addition of counterterms. The logarithmic term only comes from the bulk integral and not from the boundary.

Having obtained the renormalised on-shell action we now wish to see the dependence on the choice of (arbitrary) $g^{(0)}$. To do so we look at the variation with respect to a conformal transformation

$$\delta g^{(0)} = 2g^{(0)}\delta\sigma \; . \tag{1.53}$$

The variation takes the form

$$\delta\mathcal{L}_{finite} = -\int_{\partial X_{d+1}} \mathrm{d}^d x \sqrt{\det g^{(0)}} \delta\sigma \ \mathcal{A} \tag{1.54}$$

with $\mathcal{A}$ the anomaly. For d odd the anomaly vanishes, whilst for d even it is given by

$$\mathcal{A} = \frac{1}{16\pi G_N^{(d+1)}}(-2a^{(d)}) \ . \tag{1.55}$$

This can be seen from noting the transformation of the logarithmic term under the scaling of ϵ. On general grounds the anomaly must be of the form

$$a^{(d)} = dl^{d-1}(E^{(d)} + I^{(d)}) \tag{1.56}$$

with $E^{(d)}$ proportional to the Euler density and $I^{(d)}$ a conformal invariant.

For $2d$ one can easily compute

$$a^{(2)} = l\mathrm{Tr}((g^{(0)})^{-1}g^{(2)}) \tag{1.57}$$

and the anomaly becomes

$$\mathcal{A} = -\frac{c}{24\pi}R \ , \tag{1.58}$$

with

$$c = \frac{3l}{2G_N^{(3)}} \ . \tag{1.59}$$

One sees that in 2d the relevant object to compute for the central charge is the 3d Newton's constant. In fact this is a general statement for all dimensions. For the case of the 4d field theories dual to the 5d Sasaki-Einstein solutions we discuss in the next section the (inverse) Newton's constant is related to the Riemannian volume of the manifold. In 2d the relevant object to compute is instead a warped volume of the internal manifold and we discuss this further in appendix B.6. Later in this thesis we shall compute this *central charge c* from both the field theory and gravity of our dual pairs and find agreement.

1.2.2 Sasaki-Einstein solutions

In the twenty years since the advent of AdS/CFT many more examples of dual pairs have been proposed. One such family of such solutions follows from noticing that the space transverse to the D3-branes in the example above was $\mathbb{R}^6$ which may be written as a (Ricci-flat) cone over S^5. A natural extension is to probe the D3 branes instead by a different Calabi–Yau cone, that is the transverse space $\mathbb{R}^6$ from before is replaced with a different Calabi–Yau cone

$$\mathrm{d}s^2(Y) = \mathrm{d}r^2 + r^2\mathrm{d}s^2(X) \ , \tag{1.60}$$

where X is by definition a Sasaki-Einstein manifold. The dual field theories are 4d $\mathcal{N}=1$ quiver theories[7]. Much progress has been made on the duality, particularly in the case when the Sasaki-Einstein manifold is toric, that is the cone may be written as T^3 fibration over a 3d polytope, [28–30].

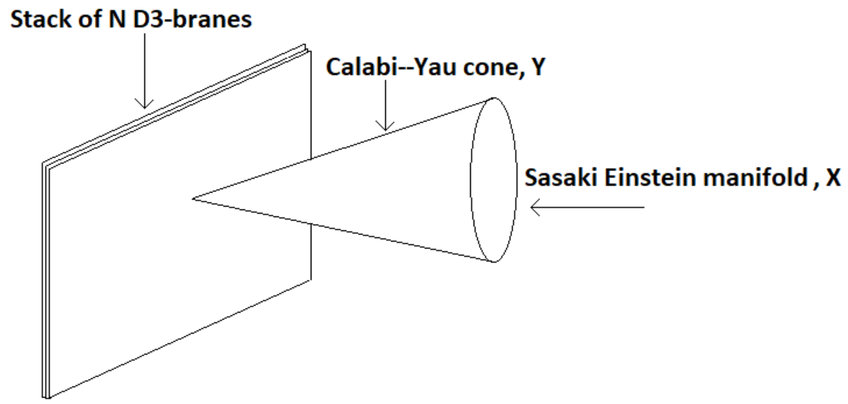

Figure 1.2: The Calabi–Yau cone probes the stack of N D3-branes.

One such dual pair, that will reappear later in this thesis, is known as $Y^{p,q}$. The metric on this space was found in [31, 32] and the field theory was first discussed in [28].

The Sasaki-Einstein metric is given by

$$\begin{aligned} \mathrm{d}s^2 =& \frac{1-y}{6}(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)+\frac{1}{w(y)q(y)}\mathrm{d}y^2+\frac{q(y)}{9}(\mathrm{d}\psi-\cos\theta\mathrm{d}\phi)^2 \\ &+w(y)\left(\mathrm{d}\alpha+\frac{a-2y+y^2}{6(a-y^2)}(\mathrm{d}\psi-\cos\theta\mathrm{d}\phi)\right)^2 \end{aligned} \tag{1.61}$$

with

$$w(y)=\frac{2(a-y^2)}{1-y}\ , \quad q(y)=\frac{a-3y^2+2y^3}{a-y^2}\ . \tag{1.62}$$

It is Einstein with $R_{\mu\nu}=4g_{\mu\nu}$ and topologically $S^2\times S^3$. The metric possesses an explicit $SU(2)\times U(1)\times U(1)$ isometry, the latter of which is identified with the R-symmetry as it acts on the Killing spinors. The metric is labeled by two integers $p>q>0$ which are the Chern numbers of the $U(1)$ fibration in the last line of (1.61) over the two two-cycles in the geometry. From the metric one may infer the toric data defining the polytope over which the T^3 fibration gives $Y^{p,q}$,

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\ ,\quad \begin{pmatrix}1\\1\\0\end{pmatrix}\ ,\quad \begin{pmatrix}1\\p\\p\end{pmatrix}\ ,\quad \begin{pmatrix}1\\p-q-1\\p-q\end{pmatrix}\ . \tag{1.63}$$

[7]The theories have been shown to be quiver theories when the cone admits a resolution. Note that for a general Sasaki-Einstein manifold (except S^5) the point $r=0$ is singular and should therefore be resolved.

Progress on identifying the dual field theory followed from a theorem by Delzant which gives a gauged linear sigma model for the cone $C(Y^{p,q})$. There is an algorithm from which one can obtain the charges of the gauged linear sigma model [33] resulting in a $U(1)$ theory with four chiral superfields with charges $(p, pq-p, -(p+q))$. The number of gauge groups is determined from the geometry to be $2p$ and the number of chiral fields is $4p+2q$. The chiral fields are in the bifundamental representation grouped into the four types of fields denoted Y, Z, U_α, V_α. The field theory has $SU(2)_1 \times U(1)_2 \times U(1)_B \times U(1)_R$ symmetry. The first two are flavour symmetries, the third is baryonic and the last is R-symmetry. For the convenience of the reader, the charges of the fields, together with their multiplicities, are summarised in Table 1.2.2.

Fields	Multiplicity	$U(1)_1$	$U(1)_2$	$U(1)_B$	$U(1)_R$
Y	$p+q$	0	-1	$p-q$	R_Y
Z	$p-q$	0	1	$p+q$	R_Z
U_1	p	1	0	$-p$	R_U
U_2	p	-1	0	$-p$	R_U
V_1	q	1	1	q	R_V
V_2	q	-1	1	q	R_V
λ	$2p$	0	0	0	1

Table 1.2: The charges of the various fields in the 4d $Y^{p,q}$ theories.

1.3 F-theory meets Holography

Recall that in section 1.1.4 the spectrum was computed only in the abelian theory, $N=1$. In order to perform the spectrum computation for the non-abelian theory ($N>1$) it was necessary for the authors of [16] to approach the problem from reducing the 6d M5-brane theory on an elliptic surface with base C. The varying coupling is then geometric data and S-duality is the symmetry of the elliptic fibration, the topological duality twist then becomes the usual topological twist. Performing the analogous non-abelian computation directly from the 4d theory has not been performed. Difficulties arise due to the lack of understanding of S-duality in the non-abelian theory. Strictly the $U(1)_D$ symmetry does not survive the non-abelian generalisation, but in the large N limit it emerges again [34]. This therefore looks ripe for using holographic methods.

This is not a marriage without its problems. For a start the two speak different languages; the language of F-theory is algebraic geometry whilst that of holography is differential geometry. In holography one generally needs a smooth[8] metric in

[8]Singularities are allowed when they can be understood physically, such as the existence of a

order to perform computations. Typically in F-theory such a metric is not known explicitly[9] and necessarily has singularities. Recall from our discussion above that a non-singular τ, which is pleasing to a holographer, implies a trivial fibration which is displeasing to an F-theorist.

Nevertheless as we shall argue during this thesis these problems are not *causa repudii*, and by using the tools of both we are able to make some non-trivial predictions between the field theory and gravity.

We should remark on other supergravity solutions with holographic duals, where non-trivial profiles of the axio-dilaton have appeared – however none of which include a varying axio-dilaton with the full $SL(2, \mathbb{Z})$ monodromy, which we incorporate in this paper. AdS-duals with particular constant, but not necessarily perturbative, values of the axio-dilaton, which correspond to F-theory at constant coupling were studied in [35–37].

There are also holographic setups with D3- and D7-branes, where the latter play the role of flavour symmetries in the field theory dual, see *e.g.* [38] for a review. Typically these correspond to configurations of D3- and D7-branes sharing four flat spacetime directions, corresponding to non-conformal four-dimensional field theories. When the backreaction of the D7-branes is included, the supergravity solutions do not have an AdS factor. Another closely related setup involving D3- and D7-branes was discussed in [39–41]. These are configurations where, as in the present paper, the branes share two dimensions. Here the D3-branes are placed into the supergravity background sourced by the D7-branes, however the Type IIB solution does not possess $SO(2, 2)$ isometry, and therefore it is not holographically dual to a 2d SCFT. This is distinct from the setup that we consider, in that it corresponds to a 4d gauge theory in the presence of 2d defects.

Recently, AdS$_6$ solutions dual to 5d SCFTs were constructed in Type IIB supergravity, which have a non-trivial τ profile that allows for poles in τ, but does not include any $SL(2, \mathbb{Z})$ monodromy [42–44]. Furthermore there is the class of holographic duals to Janus configurations, [45–48] where the gauge coupling varies along a real line, which was later generalised to the θ-angle varying along the 1d line [49]. In contrast, in our configurations, the complexified coupling τ varies holomorphically along the base of the fibration, which is a complex surface in the present case, giving rise to an elliptic fibration with general $SL(2, \mathbb{Z})$ monodromy.

certain brane.

[9]In the case of compact Calabi–Yau's, no explicit smooth metric, except the n-Torus, are known, though they have been shown to exist by Yau's theorem.

Chapter 2

General conditions for AdS_3 geometries admitting F-theory interpretations

2.1 Introduction

Twenty years after holography was uncovered in string theory, it still provides us with surprising and deep results about strongly coupled superconformal field theories (SCFTs) and quantum gravity in anti-de Sitter (AdS) spacetimes. Progress is as far ranging as finding new supergravity solutions, matching with dual field theory observables, as well as performing precision tests of the duality in particular regimes.

As reviewed in the introductory chapter, F-theory has a firm standing as a framework for constructing Type IIB Minkowski vacua in even dimensions, which preserve minimal supersymmetry. The main focus thus far in utilising F-theory has been on the construction and classification of Type IIB, Minkowski vacua with varying axio-dilaton τ, as well as (p, q) 7-branes, which are naturally encoded in the singularities of τ. The canonical setup to construct such vacua is the compactification on elliptic Calabi–Yau varieties Y_d of complex dimension d with base B_{d-1}, where the complex structure of the elliptic fiber models the axio-dilaton.

In this chapter we will discuss expanding the AdS/CFT dictionary towards theories with spacetime varying coupling constant. The main goal is the construction of Type IIB solutions including an AdS factor, where the axio-dilaton τ varies over parts of spacetime, including monodromies in the $SL(2, \mathbb{Z})$ duality group of Type IIB. In this sense these are AdS solutions in F-theory [2]. In a brane realisation, the non-trivial monodromies arise through the presence of non-perturbative (p, q) 7-branes, which contribute a new sector to the field theory duals.

The motivation to study these backgrounds is wide-ranging. On one hand it is an interesting problem in itself to classify all the possible backgrounds which admit supersymmetric AdS factors. On the other hand such understanding of the geometry allows for non-trivial predictions on the field theory side allowing us to probe non-trivial regimes of the dual field theories otherwise inaccessible to current field theory methods. Often the field theory side is somewhat difficult to study due to genuinely non-perturbative effects; by appealing to holography some of these effects may be understood. For example, in F-theory D3-branes wrapped on cycles inside the compactification geometry give rise to a varying complexified coupling τ. A field theoretic description of these is available for abelian theories [15, 16, 18, 19], but remains elusive for the non-abelian generalization. Some special cases of S-duality twists can be studied along the lines of [50], but do not correspond to varying axio-dilaton configurations. Our holographic results allow us to make some non-trivial predictions about the dual field theories.

In particular there has been recent interest[1] in various D3-brane configurations within F-theory which have been shown to give rise to 2d SCFTs [18, 19, 52–54]. These constructions are based on D3-branes wrapped on a complex curve C in the base B_{d-1} of the elliptic fibration. Our goal here is to construct holographic duals to such 2d SCFTs.

Holographically the constant axio-dilaton case supported by only five-form flux was studied in [55], where it was shown that the internal space locally admits a circle fibration over a warped Kähler base. A related analysis appeared in [56], which again has trivial τ but allows for a particular three-form flux on the internal manifold $\mathcal{M}_7$. Examples of solutions were obtained in [57, 58], again for constant τ, where starting with the general framework of [55], the 6d Kähler base is assumed to be a direct product $C_g \times \mathcal{M}_4$, with C_g a genus-g constant curvature Riemann surface, and $\mathcal{M}_4$ a locally Kähler space equipped with a metric admitting an $SU(2) \times U(1)$ isometry. In this chapter we shall generalise these results by including a non-trivially varying axio-dilaton.

The content of this chapter is obtained from the two papers [59] and [60].

2.2 AdS$_3$ Solutions in F-theory dual to 2d $\mathcal{N} = (0, 2)$ SCFTs

The starting point of our analysis is a comprehensive exploration of the conditions of Type IIB/F-theory supergravity which yield AdS$_3$ solutions with at least 2d $(0, 2)$ supersymmetry and vanishing three-form fluxes. The main difference to earlier results in [55] is that we allow the axio-dilaton τ to have a non-trivial dependence on spacetime. The requirement for 2d $(0, 2)$ supersymmetry leads us to find new classes of solutions dual to $(0, 2)$ SCFTs and a unique family of $(0, 4)$ preserving solutions. The geometry is completely determined in terms of a "master equation"

[1]D3-branes wrapped on curves in compact Calabi–Yau threefolds were studied much earlier [51], however in those setups the coupling of the D3-brane remains constant.

(2.36), which constrains the internal geometry. This potentially yields more solutions than those found here and we leave this for the future. This equation has a reformulation in terms of an F-theoretic setting, where the axio-dilaton becomes part of the compactification geometry.

2.2.1 AdS$_3$ Ansatz and $(0,2)$ Supersymmetry

In this section we consider the most general class of bosonic, Type IIB supergravity solutions with vanishing three-form flux $G = 0$, preserving $SO(2,2)$ symmetry and at least $(0,2)$ supersymmetry. These are the most general solutions holographically dual to 2d SCFTs with $U(1)_R$ R-symmetry, realised with D3-branes and 7-branes. Including the three-form flux is possible and is the topic of as of yet unpublished work by the author and collaborators. As in [55] the 10d metric will be taken in Einstein frame to be a warped product of the form[2]

$$\mathrm{d}s^2 = \mathrm{e}^{2H}\left(\mathrm{d}s^2(\mathrm{AdS}_3) + \mathrm{d}s^2(\mathcal{M}_7)\right) \ , \tag{2.1}$$

where $\mathrm{d}s^2(\mathrm{AdS}_3)$ is the metric on AdS$_3$, with Ricci tensor

$$R_{\mu\nu} = -2m^2 g_{\mu\nu} \tag{2.2}$$

and $\mathrm{d}s^2(\mathcal{M}_7)$ is the metric on an arbitrary internal seven-dimensional manifold $\mathcal{M}_7$. We take $H \in \Omega^{(0)}(\mathcal{M}_7, \mathbb{R})$, $P \in \Omega^{(1)}(\mathcal{M}_7, \mathbb{C})$, $\tau \in \Omega^{(0)}(\mathcal{M}_7, \mathbb{C})$ and the five-form flux to be of the form

$$F^{(5)} = (1 + *)\mathrm{dvol}(\mathrm{AdS}_3) \wedge F^{(2)} \ , \tag{2.3}$$

with $F^{(2)} \in \Omega^{(2)}(\mathcal{M}_7, \mathbb{R})$ in order to preserve the $SO(2,2)$ symmetry of AdS$_3$. The Bianchi identity for $F^{(5)}$ implies

$$\mathrm{d}F^{(2)} = 0\,, \qquad \mathrm{d}\hat{*}_7 F^{(2)} = 0\,, \tag{2.4}$$

where $\hat{*}_7$ is the hodge star on the unwarped metric $\mathrm{d}s^2(\mathcal{M}_7)$. We use the spinor ansatz developed in appendix A.1

$$\epsilon = \psi_1 \otimes \mathrm{e}^{H/2}\xi_1 \otimes \theta + \psi_2 \otimes \mathrm{e}^{H/2}\xi_2 \otimes \theta\,, \tag{2.5}$$

where ψ_i are Majorana Killing spinors on AdS$_3$ and satisfy

$$\nabla_\alpha \psi_i = \frac{\alpha_i m}{2}\rho_\alpha \psi_i \ , \tag{2.6}$$

[2]For the entirety of the thesis subscripts of spaces will always indicate the real dimension.

with ρ_α the Dirac matrices with signature $(-,+,+)$. The chirality of the spinor of the dual SCFT is determined by the choice of $\alpha_i = \pm 1$, see the discussion in appendix A. The spinors ψ_i are taken to be independent Killing spinors on AdS$_3$, whilst the ξ_i are Dirac spinors on $\mathcal{M}_7$. Each independent Dirac spinor $\xi_{1,2}$ will give 2 (anti-) chiral supercharges on the boundary SCFT. To preserve $(0,2)$ supersymmetry we take ξ_2 to vanish. We shall also be interested in preserving $(2,2)$ supersymmetry in section 2.4 in which case both spinors are kept, but with opposite values for α.

The reduced supersymmetry equations for the spinors on $\mathcal{M}_7$ are obtained, [59,60] by inserting the ansatz (2.5) into the $10d$ supersymmetry equations, (1.11) and (1.12),

$$\begin{aligned} \gamma^\mu P_\mu \xi_j^c &= 0\,, & (2.7)\\ \left(\frac{1}{2}\partial_\mu H \gamma^\mu - \frac{\mathrm{i}\alpha_j m}{2} + \frac{\mathrm{e}^{-4H}}{8}\not{F}^{(2)}\right)\xi_j &= 0\,, & (2.8)\\ \left(\mathcal{D}_\mu + \frac{\mathrm{i}\alpha_j m}{2}\gamma_\mu - \frac{\mathrm{e}^{-4H}}{8}F^{(2)}_{\nu_1\nu_2}\gamma_\mu{}^{\nu_1\nu_2}\right)\xi_j &= 0\,. & (2.9)\end{aligned}$$

2.2.2 Constraints on the Geometry

In this section we investigate the torsion conditions arising from imposing the minimal amount of supersymmetry, namely $\mathcal{N} = (0,2)$ in 2d. This amount of supersymmetry is preserved by the existence of a single Dirac spinor on $\mathcal{M}_7$, and signifies that the internal 7d space admits an $SU(3)$ structure. In 7d an $SU(3)$ structure implies the existence of a real vector which foliates the space with the transverse 6d space admitting a canonical $SU(3)$ structure. In the following we show that the transverse 6d space is conformally Kähler and the existence of a supersymmetric solution is determined by a single partial differential equation, similar to the equation found in [55], for the Kähler metric on the 6d space. The remaining geometry is fixed by the choice of this Kähler metric.

Using the torsion conditions presented in appendix A.1.2 and setting $\xi_2 = 0,\ \alpha = 1$ we obtain the conditions for preserving $(0,2)$ supersymmetry. Some of the torsion conditions are trivially satisfied and we present only the non-trivial torsion conditions in the present below[3] and the general equations in appendix A.1.2. Supersymmetry implies both differential and algebraic constraints on the fluxes and bilinears.

[3] We refine the notation of the appendix for ease of reading. By setting $\xi_2 = 0$ the bilinears with a '2' index are set to zero, and it therefore becomes superfluous to keep the '11' subscript on the non-zero bilinears; apart from removing this labelling the names of the bilinears are otherwise kept the same.

The independent differential conditions satisfied by the bilinears are

$$\mathrm{d}S = 0 \tag{2.10}$$

$$\mathrm{e}^{-4H}\mathrm{d}\left(\mathrm{e}^{4H}K\right) = -2\mathrm{i}mU - \mathrm{e}^{-4H}F^{(2)}\ , \tag{2.11}$$

$$\mathrm{d}\left(\mathrm{e}^{4H}U\right) = 0\ , \tag{2.12}$$

$$\mathrm{e}^{-6H}\mathcal{D}\left(\mathrm{e}^{6H}Y\right) = 2m * Y\ , \tag{2.13}$$

$$\mathrm{e}^{-6H}\mathcal{D}\left(\mathrm{e}^{6H} * Y\right) = 0\ , \tag{2.14}$$

$$4\mathrm{d}H \wedge *Y = -\mathrm{i}\mathrm{e}^{-4H}F^{(2)} \wedge Y\ , \tag{2.15}$$

$$\mathrm{e}^{-8H}\mathrm{d}\left(\mathrm{e}^{8H} * U\right) = 2\mathrm{i}m * K\ . \tag{2.16}$$

Again, as in [59] the scalar S can be set to 1 by a constant rescaling of the Killing spinor.

To proceed we introduce an orthonormal frame for the metric and by a suitable frame rotation we may set K to be parallel to the vielbein e^7. In this frame the remaining bilinears become

$$K = -e^7\ , \tag{2.17}$$

$$U = -\mathrm{i}(e^{12} + e^{34} + e^{56})\ , \tag{2.18}$$

$$X = U \wedge K\ , \tag{2.19}$$

$$Y = (e^1 - \mathrm{i}e^2) \wedge (e^3 - \mathrm{i}e^4) \wedge (e^5 - \mathrm{i}e^6)\ . \tag{2.20}$$

A 2d SCFT with $\mathcal{N} = (0,2)$ supersymmetry has a $U(1)_R$ R-symmetry, which by the AdS/CFT dictionary is dual on the gravity side to a Killing vector generating a $U(1)$ isometry of the full solution. From the torsion conditions it follows that K defines such a Killing vector and thus is identified with the R-symmetry of the putative dual SCFT. Additional evidence is provided by computing the spinorial Lie derivative of the Killing spinor with respect to this isometry, see section 2.2.3. One finds that it is charged under this Killing vector, (2.49). The Killing spinors are only charged under the R-symmetry and this fixes K to be dual to the R-symmetry. It is useful to introduce coordinates adapted to this Killing vector (and dual one-form)

$$K^{\#} = 2m\partial_\psi\ , \qquad K = \frac{1}{2m}(\mathrm{d}\psi + \rho)\ , \tag{2.21}$$

so that the 7d metric can be written as follows

$$\mathrm{d}s^2 = \frac{1}{4m^2}(\mathrm{d}\psi + \rho)^2 + \mathrm{d}s^2(\mathcal{M}_6)\ . \tag{2.22}$$

Observe from (2.12) that the bilinear U is conformally closed and this motivates us to define the following conformally rescaled forms

$$J = \mathrm{i}m^2 \mathrm{e}^{4H} U\,, \qquad \Omega = m^3 \mathrm{e}^{6H} Y\,. \tag{2.23}$$

These new forms define a canonical $SU(3)$ structure on $\widetilde{\mathcal{M}}_6$ whose metric is conformally related to $\mathcal{M}_6$ by

$$\mathrm{d}s^2(\mathcal{M}_6) = \frac{\mathrm{e}^{-4H}}{m^2}\mathrm{d}s^2(\widetilde{\mathcal{M}}_6)\ . \tag{2.24}$$

They satisfy the $SU(3)$ structure algebraic conditions

$$J \wedge \Omega = 0\,, \qquad \Omega \wedge \bar{\Omega} = -\frac{8\mathrm{i}}{6} J \wedge J \wedge J = -8\mathrm{i}\ \mathrm{dvol}(\widetilde{\mathcal{M}}_6) \tag{2.25}$$

and in addition the differential conditions

$$\bar{\mathcal{D}}\Omega = -2\mathrm{i}mK \wedge \Omega\,, \qquad \mathrm{d}J = 0\,, \tag{2.26}$$

which imply integrability of the complex structure defined by Ω and that $\widetilde{\mathcal{M}}_6$ is Kähler. Finally, we should extract the conditions of the varying axio-dilaton on the metric. From the supersymmetry equation (2.7)

$$J^\mu{}_\nu P^\nu = \mathrm{i}P^\mu\,, \qquad P_\mu K^\mu = 0\,, \tag{2.27}$$

i.e. P is a $(1,0)$ form on $\widetilde{\mathcal{M}}_6$ and the Killing vector K is a symmetry of τ; $\mathcal{L}_K\tau = 0$.

Due to the foliation of the space by the Killing vector we may decompose the exterior derivative as

$$\mathrm{d} = \mathrm{d}\psi \wedge \partial_\psi + \mathrm{d}_6\ . \tag{2.28}$$

With this splitting of the exterior derivative (2.26) becomes

$$\partial_\psi \Omega = -\mathrm{i}\Omega\ , \tag{2.29}$$

$$\mathrm{d}_6\Omega = -\mathrm{i}(Q+\rho)\wedge\Omega\ . \tag{2.30}$$

Equation (2.29) may be solved by extracting a suitable ψ dependent phase from Ω. This phase will play no role in the following analysis and will be assumed to have been extracted. Subsequently, (2.30) implies

$$\mathfrak{R} = -(\mathrm{d}Q + \mathrm{d}\rho)\ , \tag{2.31}$$

where $\mathfrak{R}$ is the Ricci form on $\widetilde{\mathcal{M}}_6$. This implies that the curvature of the base is given by minus the curvature of the R-symmetry bundle and the curvature of the duality line bundle $\mathcal{L}_D$ with connection (1.8). In practice this equation will be used to fix the connection term ρ in (2.21) of the circle-bundle.

The Ricci tensor on $\widetilde{\mathcal{M}}_6$ is given in terms of the Ricci form as

$$R_{\mu\nu} = -J_\mu{}^\rho \mathfrak{R}_{\rho\nu} \ . \tag{2.32}$$

The flux is fixed by equation (2.11) to be

$$mF^{(2)} = -2J - \frac{1}{2}\mathrm{d}(\mathrm{e}^{4H}(\mathrm{d}\psi + \rho)) \ . \tag{2.33}$$

Notice that the flux has legs along the Killing direction and may be decomposed such that

$$\hat{F}^{(2)} = F^{(2)} + \mathrm{d}\mathrm{e}^{4H} \wedge K \ , \tag{2.34}$$

has no legs along the Killing direction[4]. By contracting the indices of the Ricci-form with the complex structure one finds the Ricci scalar for $\widetilde{\mathcal{M}}_6$ to be[5]

$$R = 2|P|^2 + 8\mathrm{e}^{-4H} \ . \tag{2.35}$$

By imposing equations (2.11), (2.26), (2.31) it follows that equations (2.14)-(2.16) are immediately satisfied.

2.2.3 Sufficiency of the Conditions

So far supersymmetry has implied that the solution satisfies (2.26), (2.27), (2.31), (2.33) and (2.35). We show in this section that this set of equations in addition to imposing the equation of motion for $F^{(2)}$ are both necessary and sufficient conditions for a bosonic supersymmetric solution. As we show, the equation of motion for $F^{(2)}$ may be rephrased as a differential condition on the Kähler metric of the 6d space. We proceed by first considering the equations of motion before proving that there exists a globally defined Killing spinor satsifying the Killing spinor equations (2.7)-(2.9).

Equations of Motion

Recall that the equation of motion for the five-form flux is equivalent to the two equations in (2.4) for the two-form $F^{(2)}$. Using equation (2.33) as the definition of $F^{(2)}$ it is clear after using (2.26) that it is closed. Supersymmetry, however does not impose the equation of motion for the flux, $\mathrm{d} * F^{(2)} = 0$, which must be imposed in addition. One may understand this equation as giving a "master equation" for the Kähler base which generalises the one found in [55] to include varying axio-dilaton,

$$\Box_6(R - 2|P|^2) - \frac{1}{2}R^2 + R_{\mu\nu}R^{\mu\nu} + 2|P|^2 R - 4R_{\mu\nu}P^\mu P^{*\nu} = 0 \ . \tag{2.36}$$

[4] The explicit K factor cancels out with that in $F^{(2)}$.

[5] In deriving this result it is necessary to use the algebraic equation $F^{(2)}_{\mu\nu}J^{\mu\nu} = \hat{F}^{(2)}_{\mu\nu}J^{\mu\nu} = -\frac{8}{m}$, which is obtained from the supersymmetry equation (2.7).

A discussion of its derivation is given in appendix A.1.3. We conclude that both the equation of motion for the five-form flux and the self-duality constraint are satisfied. The Bianchi identity for P is implied by construction whilst its equation of motion reduces to τ being harmonic on the Kähler manifold. As τ is holomorphic it follows that it is also harmonic and therefore the flux equations of motion and Bianchi identities are satisfied.

By using the analysis of [7] and some case dependent algebra we may show that the Einstein equation is satisfied. Integrability of the Killing spinor equations and use of the flux equations of motion and Bianchi identities implies

$$E_{MN}\Gamma^{N}\epsilon = 0 \tag{2.37}$$

where $E_{MN} = 0$ is equivalent to Einstein's equation and ϵ is the 10d Killing spinor. One may construct a null vector bilinear, $\widehat{K} \equiv \bar{\epsilon}\,\Gamma_{(1)}\epsilon$, which implies that the metric admits a frame such that it takes the form

$$\mathrm{d}s^2 = 2e^+e^- + e^a e^a \ , \tag{2.38}$$

with $\widehat{K} = e^+$ and $a = 1, .., 8$. The argument of [61] shows that the only component of E_{MN} which may be non-zero is E_{++}. For this class of solutions E_{++} lies along AdS_3 and by explicit computation one finds that the Ricci-tensor on the warped AdS_3 satisfies

$$R_{\mu\nu} = \left(-2m^2 + 8\nabla_\mu H \nabla^\mu H - \Box H\right) g_{\mu\nu} \ . \tag{2.39}$$

It follows that $E_{++} \propto g_{++}$ which therefore vanishes and we conclude supersymmetry implies the Einstein equation. We determine that all the equations of motion are satisfied by supersymmetry and equation (2.36).

Supersymmetry

We now show that any solution satisfying the necessary conditions presented above admits a globally defined Killing spinor satisfying (1.11) and (1.12). By construction it follows that any global solution to the 7d Killing spinor equations (2.7)-(2.9) may be uplifted to a global Killing spinor in 10d satisfying both (1.11) and (1.12). Preserving supersymmetry is therefore equivalent to proving that equations (2.7)-(2.9) admit a globally defined Killing spinor. We shall construct such a spinor by making use of the canonical spin^c structure that every Kähler manifold admits.

We begin by defining the notation and vielbein we shall be using in the following. Recall that the metric takes the form

$$\mathrm{d}s^2(\mathcal{M}_7) = \frac{\mathrm{e}^{-4H}}{m^2}\mathrm{d}s^2(\widetilde{\mathcal{M}}_6) + \frac{1}{4m^2}(\mathrm{d}\psi + \rho)^2 = \mathrm{d}s^2(\mathcal{M}_6) + (e^7)^2 \, , \tag{2.40}$$

where, in keeping with the frame in section 2.2.2, $e^7 = -\frac{1}{2m}(\mathrm{d}\psi + \rho)$. The flat index for the vielbein on $\widetilde{\mathcal{M}}_6$ will be taken from the middle of the Latin alphabet, i, j, k and run from $1, \ldots, 6$ whereas the curved index on $\widetilde{\mathcal{M}}_6$ will be from the middle of the Greek alphabet, μ, ν, σ, finally the seven-dimensional indices will be from the beginning of the respective alphabets. The fundamental two-form on $\mathcal{M}_6$, written in terms of the vielbein, is $j = e^{12} + e^{34} + e^{56}$, which in general is only conformally closed, whilst the closed Kähler two-form on $\widetilde{\mathcal{M}}_6$ is denoted J. They are related by

$$j = \frac{\mathrm{e}^{-4H}}{m^2} J \ . \tag{2.41}$$

On any Kähler manifold there exists a spin^c structure that admits a section η satisfying the spin^c Killing spinor equation

$$\left(\widetilde{\nabla}_\mu + \frac{\mathrm{i}}{2} \widehat{P}_\mu \right) \eta = 0 \, , \tag{2.42}$$

where $\widehat{P}$ is the one-form Ricci potential of the Kähler metric. For a 6d space, if one takes the spinor η to satisfy the projection conditions

$$\gamma_{12}\eta = \gamma_{34}\eta = \gamma_{56}\eta = -\mathrm{i}\eta \ \ , \tag{2.43}$$

it is easy to see that the term arising from the spin-connection precisely cancels the contribution from $\widehat{P}$ and therefore any constant section η, subject to the projection conditions, solves (2.42). Clearly this spinor is globally defined on $\widetilde{\mathcal{M}}_6$, and we may use it to to construct a Killing spinor satisfying the 7d supersymmetry equations. On $\widetilde{\mathcal{M}}_6$, equation (2.42) reads

$$\left(\widetilde{\mathcal{D}}_\mu - \frac{\mathrm{i}}{2} \rho_\mu \right) \eta = 0 \ . \tag{2.44}$$

The spin connection on $\mathcal{M}_7$ is found to be

$$\omega^{jk} = \widetilde{\omega}^{jk} - 2(\partial^k H e^j - \partial^j H e^k) - \frac{1}{4m} \left[\mathfrak{R}^{jk} - \mathrm{i}(P^j P^{*k} - P^{*j} P^k) \right] e^7 \ , \tag{2.45}$$

$$\omega^{7j} = \frac{1}{4m} \left[\mathfrak{R}^j{}_k e^k - \mathrm{i}(P^j P^* - P^{*j} P) \right] \ , \tag{2.46}$$

and the flux is

$$m F^{(2)} = -2m^2 \mathrm{e}^{4H} j + \frac{\mathrm{e}^{4H}}{2} (\mathfrak{R} + \mathrm{d}Q) + 4m \mathrm{e}^{4H} \mathrm{d}H \wedge e^7 \ . \tag{2.47}$$

By inserting the above spin connection, (2.31), (2.33) and (2.35) into (2.9), and computing along the Killing direction and along $\widetilde{\mathcal{M}}_6$, respectively, yields

$$
\begin{aligned}
0 &= \left(\nabla_\psi + \frac{\mathrm{i}m}{2} - \frac{\mathrm{e}^{-4H}}{8} F_{ab}\gamma_\psi{}^{ab}\right)\xi = \left(\partial_\psi - \frac{\mathrm{i}}{2}\right)\xi \\
0 &= \left(\mathcal{D}_\mu + \frac{\mathrm{i}m}{2}\gamma_\mu - \frac{\mathrm{e}^{-4H}}{8} F_{ab}\gamma_\mu{}^{ab}\right)\xi = \left(\widetilde{\mathcal{D}}_\mu - \frac{\mathrm{i}}{2}\rho_\mu\right)\xi \; . \qquad (2.48)
\end{aligned}
$$

We may solve both equations by taking the Killing spinor to be

$$
\xi = \mathrm{e}^{\frac{\mathrm{i}}{2}\psi}\eta \; . \qquad (2.49)
$$

Notice that the functional dependence on ψ is consistent with (2.29).

It remains to show that the algebraic conditions (2.7) and (2.8) are satisfied. Using the holomorphicity of P one finds that the dilatino equation, (2.7), vanishes upon application of the projection conditions (2.43). The algebraic gravitino equation becomes

$$
0 = \left(\frac{1}{2}\partial_\mu H\gamma^\mu - \frac{\mathrm{i}\alpha_j m}{2} + \frac{\mathrm{e}^{-4H}}{8}\not{F}^{(2)}\right)\xi = \left(\frac{\mathrm{i}m}{4} + \frac{1}{16m}\left(\frac{1}{2}\mathfrak{R}_{ij} - \mathrm{i}P_i P_j^*\right)\gamma^{ij}\right)\xi \, , \qquad (2.50)
$$

which vanishes after some gamma matrix algebra and application of (2.35) and (2.43). We conclude that supersymmetry is preserved if we satisfy (2.26), (2.27), (2.31), (2.33), (2.35) and (2.36).

2.2.4 Summary of Conditions

Let us summarise the necessary and sufficient conditions for a supersymmetric solution with at least $\mathcal{N} = (0, 2)$ supersymmetry, metric of the form (2.1), arbitrary five-form flux, F and varying axio-dilaton, τ all preserving the isometries of AdS_3. We have shown that the metric of the solution takes the form

$$
\mathrm{d}s^2 = \mathrm{e}^{2H}\left[\mathrm{d}s^2(\mathrm{AdS}_3) + \frac{1}{m^2}\left(\frac{1}{4}(\mathrm{d}\psi + \rho)^2 + \mathrm{e}^{-4H}\mathrm{d}s^2(\widetilde{\mathcal{M}}_6)\right)\right] \, , \qquad (2.51)
$$

where $\mathrm{d}s^2(\widetilde{\mathcal{M}}_6)$ is a Kähler metric satisfying the "master equation" (2.36). The remaining geometry is determined in terms of the metric on $\mathrm{d}s^2(\widetilde{\mathcal{M}}_6)$ to be

$$
\mathrm{e}^{-4H} = \frac{1}{8}(R - 2|P|^2) \; , \qquad (2.52)
$$

$$
\mathrm{d}\rho = -(\mathrm{d}Q + \mathfrak{R}) \; , \qquad (2.53)
$$

and the flux is given by

$$\begin{aligned} F &= (1+*)\mathrm{dvol}(\mathrm{AdS}_3)\wedge F^{(2)} \\ mF^{(2)} &= -2J - \frac{1}{2}\mathrm{d}(\mathrm{e}^{4H}(\mathrm{d}\psi+\rho)) \ . \end{aligned} \tag{2.54}$$

The axio-dilaton τ is a holomorphic function on $\widetilde{\mathcal{M}}_6$, and when it is constant, the above conditions consistently reduce to those in [55]. As shown in the previous subsection these conditions are necessary and sufficient for the existence of a supersymmetric solution.

2.2.5 F-theoretic Formulation

The condition on the curvature and axio-dilaton (2.36) has again a nice geometrized form which will allow a re-interpretation of the Type IIB supergravity equations with varying τ in terms of an F-theory model, where the axio-dilaton τ is identified with the complex structure of an elliptic curve. The varying of the complex structure, which is compatible with the $SL(2,\mathbb{Z})$ duality group action on Type IIB string theory, is then encoded in a geometric elliptic fibration in a putative 12d space.

The geometry that incorporates the axio-dilaton in terms of an elliptic fibration over the Type IIB spacetime $\widetilde{\mathcal{M}}_6$ is a Kähler four-fold, with metric

$$\mathrm{d}s^2(\mathcal{Y}_8^\tau) = \frac{1}{\tau_2}\left((\mathrm{d}x+\tau_1\mathrm{d}y)^2+\tau_2^2\mathrm{d}y^2\right)+\mathrm{d}s^2(\widetilde{\mathcal{M}}_6) \ , \tag{2.55}$$

whose Ricci-form is written in terms of that of $\widetilde{\mathcal{M}}_6$, $\mathfrak{R}^{(\widetilde{M})}$, as

$$\mathfrak{R}^{(\mathcal{Y})} = \mathfrak{R}^{(\widetilde{M})} - \mathrm{i}P\wedge P^* \ . \tag{2.56}$$

It is clear from this expression that the Ricci-form has legs only along $\widetilde{\mathcal{M}}_6$ and therefore

$$R^{(\mathcal{Y})}_{\mu\nu} = R^{(\widetilde{M})}_{\mu\nu} - 2P_{(\mu}P^*_{\nu)} \ , \tag{2.57}$$

$$R^{(\mathcal{Y})} = R^{(\widetilde{M})} - 2|P|^2 \ . \tag{2.58}$$

Using the above expressions in (2.36) and that the coordinates of the auxiliary elliptic fibration generate Killing directions of the full solution we find

$$\begin{aligned} 0 &= \Box_{\widetilde{M}}(R^{(\widetilde{M})}-2|P|^2) - \frac{1}{2}(R^{(\widetilde{M})})^2 + R^{(\widetilde{M})}_{\mu\nu}R^{(\widetilde{M})\mu\nu} + 2|P|^2R^{(\widetilde{M})} - 4R^{(\widetilde{M})}_{\mu\nu}P^\mu P^{*\nu} \\ &= \Box_{\mathcal{Y}}R^{(\mathcal{Y})} - \frac{1}{2}(R^{(\mathcal{Y})})^2 + R^{(\mathcal{Y})}_{ij}R^{(\mathcal{Y})\,ij} \ . \end{aligned} \tag{2.59}$$

This is the "master equation" presented in [55] in two more dimensions. Solving (2.36) is equivalent to solving (2.59) and imposing that the 8d Kähler metric for

$\mathcal{Y}_8^\tau$ is elliptically fibered. The condition is thus *not* that this space is Calabi-Yau, but a more refined condition, which only in special cases will be shown to reduce to containing Ricci-flat elliptic fibrations. Alternatively, the geometry may also be specified in terms of the metric on $\mathcal{Y}_8^\tau$ using (2.52) and (2.53) as

$$R^{(\mathcal{Y})} = 8\mathrm{e}^{-4H} \ , \qquad \mathrm{d}\rho = -\mathfrak{R}^{(\mathcal{Y})} \, . \tag{2.60}$$

Note that solutions to this equation will also automatically give rise to supersymmetric solutions of eleven dimensional supergravity of the form $\mathrm{AdS}_2 \times \mathcal{M}_9$ [62], where $\mathcal{M}_9$ is locally a circle fibration over $\mathcal{Y}_8^\tau$. We thus obtain a 1–1 correspondence of F-theory AdS_3 solutions and elliptically fibered M-theory AdS_2 solutions.

2.2.6 AdS_2 Solutions in M-theory

The "master equation" (2.59) is the same as the equation in [62] governing AdS_2 solutions in 11d supergravity with only electric flux. The AdS_3 F-theory solutions are a subclass of those solutions when the (real) 8d Kähler base is taken to be elliptically fibered. To perform this duality chain we must write the AdS_3 metric as a foliation by AdS_2 [57], that is we use the metric

$$\mathrm{d}s^2(\mathrm{AdS}_3) = \frac{1}{4m^2}\left(-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + (2\mathrm{d}\varphi + r\mathrm{d}t)^2\right) \, . \tag{2.61}$$

We have normalised the metric such that the Ricci-tensor satisfies $R_{\mu\nu} = -2m^2 g_{\mu\nu}$. One may now perform a T-duality along the azimuthal coordinate φ to obtain the metric on $\mathrm{AdS}_2 \times S^1$ with the full $SO(1,1)$ isometry group of AdS_2 preserved. Performing the T-duality on the general Type IIB solution given in (2.51) and (2.54) results in the string frame Type IIA solution

$$\begin{aligned}
m^2\mathrm{d}s^2(\mathcal{M}_{IIA}) &= \frac{\mathrm{e}^{2H}}{\sqrt{\tau_2}}\left(\frac{1}{4}\left(-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2}\right) + \frac{1}{4}(\mathrm{d}\chi + \rho)^2 + \mathrm{e}^{-4H}\mathrm{d}s^2(\mathcal{M}_6)\right) + \sqrt{\tau_2}\mathrm{e}^{-2H}\mathrm{d}\varphi^2 \ , \\
F_4^{IIA} &= \frac{1}{4m^2}\mathrm{dvol}(\mathrm{AdS}_2) \wedge F^{(2)} \ , \\
F_2^{IIA} &= \frac{1}{m}\mathrm{d}\tau_1 \wedge \mathrm{d}\varphi \ , \\
H &= \frac{1}{2}\mathrm{d}\varphi \wedge \mathrm{dvol}(\mathrm{AdS}_2) \ , \\
\mathrm{e}^{-2\Phi_{IIA}} &= \tau_2^{\frac{3}{2}}\mathrm{e}^{2H} \ ,
\end{aligned} \tag{2.62}$$

which uplifts to 11d supergravity as an $\text{AdS}_2 \times \mathcal{M}_9$ solution

$$
\begin{aligned}
m^2 \mathrm{d}s^2(\mathcal{M}_{11}) &= \mathrm{e}^{\frac{8H}{3}} \left(\frac{m^2}{4} \mathrm{d}s^2(\text{AdS}_2) + \frac{1}{4}(\mathrm{d}\chi + \rho)^2 \right. \\
&\quad \left. + \mathrm{e}^{-4H} \left(\mathrm{d}s^2(\mathcal{M}_6) + \tau_2 \mathrm{d}\varphi^2 + \frac{1}{\tau_2}(\mathrm{d}\psi + \tau_1 \mathrm{d}\varphi)^2 \right) \right) \,, \\
G_4 &= \frac{1}{4m} \mathrm{dvol}(\text{AdS}_2) \wedge \left[-2J_8 - \frac{1}{2} \mathrm{d}(\mathrm{e}^{4H}(\mathrm{d}\chi + \rho)) \right] \,,
\end{aligned}
\tag{2.63}
$$

which agrees with the general form presented in [62] upon making the identifications

$$
A_{KP} = \frac{4H}{3} \,, \qquad B_{KP} = \frac{1}{2}\rho \,, \qquad \psi_{KP} = \frac{\chi}{2} \,. \tag{2.64}
$$

Observe that the elliptically fibered space $\mathcal{Y}_8^\tau$, that underlies the F-theory solutions of section 2.2.5, now appears in the solution explicitly as part of the geometry.

2.3 Susy, susy everywhere; the $\mathcal{N} = (0, 4)$ story

Finding solutions of the above general conditions is in general a difficult undertaking. A technique to simplify the problem is that of imposing the existence of more supersymmetry. This typically leads to simpler equations as the solution is more constrained. Before delving into finding solutions of (2.36) we shall investigate the conditions arising from imposing a greater amount of supersymmetry. There are two options; requiring $(0, 4)$ supersymmetry which will be the content of this section or $(2, 2)$ supersymmetry which is the content of the preceding section.

We find that the most general solutions in this class admit an $SU(2)$ structure. In seven-dimensions an $SU(2)$ structure implies the existence of three independent one-forms orthogonal to a four-dimensional foliation with $SU(2)$ structure. This is specified by a real two-form of maximal rank and a complex two-form satisfying the $SU(2)$ structure relations (provided later) which are the $SU(2)$ analogue of the $SU(3)$ relations (2.25). The G-invariant tensors obtained from the Killing spinors are defined in appendix A.1.2. To compute the algebraic relations imposed by the $SU(2)$ structure we shall introduce an orthonormal frame using the gamma matrices defined in appendix A.1. One may recover these results by making use of Fierz identities, the two methods are equivalent.

In the following we summarise the results, with more details provided in appendix A.1.2. We specialise those equations to the relevant case $\alpha_1 = \alpha_2 = 1$.

From (A.45) and (A.61) we obtain the following conditions on the scalar bilinears

$$
S_{11} = S_{22} \;=\; 1 \,, \tag{2.65}
$$

$$
A_{11} = A_{22} = A_{12} = S_{12} \;=\; 0 \,. \tag{2.66}
$$

From (A.60) we see that there are three independent Killing vectors. Imposing that the Killing vectors lie along a subspace defined by the vielbeins $e^5,\ e^6,\ e^7$, consistent with an $SU(2)$ structure, is equivalent to imposing the projection condition

$$\gamma_{1234}\xi_i = -\xi_i\,. \tag{2.67}$$

In addition we have the freedom to choose K_{11} to be parallel to e^7. In this frame the independent one-forms and two-forms are given by

$$\begin{aligned}
K_{11} &= -K_{22} = e^7\,, && (2.68)\\
K_{12} &= e^5 - \mathrm{i}e^6\,, && (2.69)\\
B_{12} &= 0\,, && (2.70)\\
U_{11} &= -\mathrm{i}(e^{12} + e^{34} - e^{56})\,, && (2.71)\\
U_{22} &= -\mathrm{i}(e^{12} + e^{34} + e^{56})\,, && (2.72)\\
V_{11} &= V_{22} = 0\,, && (2.73)\\
V_{12} &= -(e^1 - \mathrm{i}e^2)\wedge(e^3 - \mathrm{i}e^4)\,. && (2.74)
\end{aligned}$$

The remaining forms may be expressed in terms of the forms defined above as

$$\begin{aligned}
U_{12} =& K_{11}\wedge K_{12}\,, && (2.75)\\
X_{11} =& U_{11}\wedge K_{11}\,, && (2.76)\\
X_{22} =& U_{22}\wedge K_{22}\,, && (2.77)\\
X_{12} =& U_{11}\wedge K_{12} = U_{22}\wedge K_{12}\,, && (2.78)\\
Y_{11} =& V_{12}\wedge K_{12}^*\,, && (2.79)\\
Y_{22} =& -V_{12}\wedge K_{12}\,, && (2.80)\\
Y_{12} =& -V_{12}\wedge K_{11} = V_{12}\wedge K_{22}\,. && (2.81)
\end{aligned}$$

2.3.1 Reducing the torsion conditions

After introducing the frame, it is now possible to reduce the differential conditions to a minimal set. The remaining non-trivial conditions are

$$\begin{aligned}
\mathrm{e}^{-4H}\mathrm{d}\left(\mathrm{e}^{4H}K_{jj}\right) &= -2\mathrm{i}mU_{jj} - \mathrm{e}^{-4H}F^{(2)}\,, && (2.82)\\
\mathrm{e}^{-4H}\mathrm{d}\left(\mathrm{e}^{4H}K_{12}\right) &= -2\mathrm{i}mU_{12}\,, && (2.83)\\
\mathrm{d}\left(\mathrm{e}^{4H}U_{ij}\right) &= 0\,, && (2.84)\\
\mathcal{D}\left(\mathrm{e}^{6H}V_{12}\right) &= 0\,. && (2.85)
\end{aligned}$$

The frame computation implies $K_{11} = -K_{22} = e^7$, and inserting this into (2.82) gives an expression for $F^{(2)}$

$$F^{(2)} = -\mathrm{i}m\mathrm{e}^{4H}(U_{11}+U_{22}) = -2m\mathrm{e}^{4H}\left(e^{12}+e^{34}\right) . \tag{2.86}$$

Notice that (2.84) implies that (2.86) satisfies the Bianchi identity for $F^{(2)}$. From this explicit expression we may also compute $*F^{(2)}$ and show that it satisfies its equation of motion. Observe from the algebraic equation (2.8) we have the relation

$$\partial_\mu H = -\frac{\mathrm{e}^{-4H}}{4} F_{\mu\nu}\bar{\xi}_1\gamma^\nu\xi_1 , \tag{2.87}$$

which together with (2.86) implies that the warp factor is constant

$$\mathrm{d}H = 0 . \tag{2.88}$$

Next we can determine the Killing vectors. From (A.60) we see that there are three independent Killing vectors of the *full* solution whose dual one-forms are

$$K_{11} = -K_{22} = e^7 , \tag{2.89}$$

$$\mathrm{Re}\,[K_{12}] = e^5 , \tag{2.90}$$

$$\mathrm{Im}\,[K_{12}] = -e^6 . \tag{2.91}$$

From the torsion conditions (2.82) the dual one-forms to these Killing vectors satisfy the differential conditions

$$\mathrm{d}e^5 = 2me^{67} , \tag{2.92}$$

$$\mathrm{d}e^6 = 2me^{75} , \tag{2.93}$$

$$\mathrm{d}e^7 = 2me^{56} , \tag{2.94}$$

which is a warped form of the equations obeyed by the $SU(2)$ invariant one-forms [6].

[6] The Maurer–Cartan left-invariant one-forms in the coordinates we are using are

$$\sigma_{1,L} = -\sin\psi\mathrm{d}\theta + \cos\psi\sin\theta\mathrm{d}\varphi , \quad \sigma_{2,L} = \cos\psi\mathrm{d}\theta + \sin\psi\sin\theta\mathrm{d}\varphi , \quad \sigma_{3,L} = \mathrm{d}\psi + \cos\theta\mathrm{d}\varphi .$$

The coordinates have periods $\psi \in [0,4\pi]$, $\varphi \in [0,2\pi]$, $\theta \in [0,\pi]$ .These satisfy $\mathrm{d}\sigma_{i,L} = \frac{1}{2}\epsilon_{ijk}\sigma_{j,L}\wedge\sigma_{k,L}$. The right-invariant one-forms we shall take are

$$\sigma_{1,R} = \sin\varphi\mathrm{d}\theta - \cos\varphi\sin\theta\mathrm{d}\psi , \quad \sigma_{2,R} = \cos\varphi\mathrm{d}\theta + \sin\varphi\sin\theta\mathrm{d}\psi , \quad \sigma_{3,R} = \mathrm{d}\varphi + \cos\theta\mathrm{d}\psi .$$

These satisfy $\mathrm{d}\sigma_{i,R} = -\frac{1}{2}\epsilon_{ijk}\sigma_{j,R}\sigma_{k,R}$. Of course we could take the right-invariant one-forms to solve the same equation as the left-invariant one-forms however in the present case we have the desirable property $\sigma_{1,L}\wedge\sigma_{2,L}\wedge\sigma_{3,L} = \sigma_{1,R}\wedge\sigma_{2,R}\wedge\sigma_{3,R}$.

On the 4d subspace B we have an $SU(2)$ structure with the Kähler-form given by

$$J_B = \frac{\mathrm{i}}{2}\left(U_{11} + U_{22}\right) = e^{12} + e^{34}\,, \tag{2.95}$$

with corresponding holomorphic two-form satisfying $\frac{1}{2}\Omega_B \wedge \bar{\Omega}_B = J_B \wedge J_B = 2\mathrm{dvol}(B)$, which takes the form

$$\Omega_B = -V^*_{12} = \left(e^1 + \mathrm{i}e^2\right) \wedge \left(e^3 + \mathrm{i}e^4\right)\,. \tag{2.96}$$

With these definitions the remaining torsion conditions become

$$\mathrm{d}J_B \;=\; 0\,, \tag{2.97}$$

$$\bar{D}\Omega_B \;=\; 0\,. \tag{2.98}$$

Furthermore, from (A.63) and (A.64) it follows that P may only have components along B and using (2.7) we find

$$J_{Bm}{}^{n} P_n = \mathrm{i}P_m\,, \tag{2.99}$$

and hence P is a $(1,0)$ form. The form of P then implies that τ is holomorphic and therefore satisfies

$$\mathrm{d}\tau \wedge \Omega_B = 0\,. \tag{2.100}$$

Notice that (2.99) implies the necessary conditions

$$P^2 = 0\,, \quad \Box\,\tau = 0\,. \tag{2.101}$$

which implies the equation of motion for P. From (2.98) we may identify $-Q$ as the canonical Ricci-form potential on the Kähler manifold B and hence we have

$$\mathfrak{R} + \mathrm{d}Q = 0\,, \tag{2.102}$$

where $\mathfrak{R}$ is the Ricci-form on B. Making use of the identity,

$$\mathfrak{R}^{m_1}{}_{m_2} = J^{m_1}{}_{n} R^{n}{}_{m_2}\,, \tag{2.103}$$

the condition in (2.102) may be expressed as

$$R^{Y_3}_{mn} \equiv R_{mn} - \frac{1}{2\tau_2^2}\left(\partial_m\tau_1\partial_n\tau_1 + \partial_m\tau_2\partial_n\tau_2\right) = 0\,. \tag{2.104}$$

This equation relates the Ricci tensor of the base to the variation of τ over B. In particular, this is the Ricci flatness condition for the metric of an elliptically fibered Calabi–Yau threefold Y_3 valid away from the singularities in the fiber.

Since τ is holomorphic, away from loci where the fiber degenerates, the metric for the elliptically fibered Calabi–Yau can be written as

$$\mathrm{d}s^2(Y_3) = \frac{1}{\tau_2}\left((\mathrm{d}x + \tau_1 \mathrm{d}y)^2 + \tau_2^2 \mathrm{d}y^2\right) + \mathrm{d}s^2(B)\,. \tag{2.105}$$

Indeed, imposing that this metric is Ricci flat implies that the Ricci tensor on B satisfies (2.104). As was noted in [12] this local metric is singular over the discriminant locus of the elliptic fibration.

To exhibit the Calabi–Yau condition, we construct the Kähler form and holomorphic three-form of the Calabi–Yau threefold from the corresponding quantities of the base, which define an $SU(3)$ structure. Let the vielbein on the fibration be

$$e^1 = \frac{1}{\sqrt{\tau_2}}(\mathrm{d}x + \tau_1 \mathrm{d}y)\,, \quad e^2 = \sqrt{\tau_2}\mathrm{d}y\,, \tag{2.106}$$

then

$$J_{Y_3} = e^{12} + J_B\,, \qquad \Omega_{Y_3} = (e^1 + \mathrm{i}e^2)\wedge \Omega_B\,. \tag{2.107}$$

With this frame on the elliptic fibration and giving indices $3, 4, 5, 6$ to the base, some relevant components of the spin connection which are useful later on are

$$\omega^1{}_2 = Q, \qquad \omega^3{}_4 + \omega^5{}_6 = -Q\,, \qquad \omega^4{}_5 + \omega^3{}_6 = 0\,, \qquad \omega^3{}_5 - \omega^4{}_6 = 0\,. \tag{2.108}$$

The Calabi–Yau condition in terms of this $SU(3)$ structure is equivalent to

$$\mathrm{d}J_{Y_3} = 0\,, \qquad \mathrm{d}\Omega_{Y_3} = 0\,. \tag{2.109}$$

Upon using $\mathrm{d}J_B = 0$ it follows trivially that $\mathrm{d}J_{Y_3} = 0$. Consider instead $\mathrm{d}\Omega_{Y_3} = 0$; we have

$$\begin{aligned}\mathrm{d}\Omega_{Y_3} &= \left(-\frac{1}{2\tau_2}\mathrm{d}\tau_2 - \mathrm{i}Q\right)\wedge \Omega_{Y_3} + \frac{1}{\tau_2}\mathrm{d}\tau\wedge \mathrm{d}y \wedge \Omega_B \\ &= \frac{\mathrm{i}}{2\tau_2}\mathrm{d}\tau \wedge \Omega_{Y_3} + \frac{1}{\tau_2}\mathrm{d}\tau\wedge\mathrm{d}y\wedge\Omega_B\,,\end{aligned} \tag{2.110}$$

where we have used (2.98) in the first line. Upon using the holomorphicity of τ and that it depends only on the base coordinates this is identically zero. This shows that (2.98) and therefore also (2.102) and (2.104) are equivalent to B being the base of an elliptically fibered Calabi–Yau threefold. It is then easy to see that the Einstein condition is satisfied.

In summary, the solution

$$\begin{array}{c} \mathbb{E}_\tau \\ \downarrow \\ \mathrm{AdS}_3 \times S^3 \times B \end{array} \tag{2.111}$$

where the elliptic fibration over B gives rise to a Calabi–Yau threefold, is thus given by[7]

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathrm{AdS}_3) + \frac{1}{4m^2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) + \frac{1}{m_B^2}\mathrm{d}s^2(B)\,, \tag{2.112}$$

$$F = -(1+*)\frac{2m\mathrm{e}^{4H}}{m_B^2}J_B \wedge \mathrm{dvol}(\mathrm{AdS}_3)\,, \tag{2.113}$$

$$P = \frac{\mathrm{i}}{2\tau_2}\mathrm{d}\tau\,, \tag{2.114}$$

where J_B is the Kähler form on the base of the elliptically fibered Calabi–Yau threefold, and τ varies holomorphically over B. Here, $1/m_B$ is the length scale associated to the base B.

The possible base manifolds of an elliptically fibered Calabi–Yau threefold were determined in [63, 64] and found to be one of the following: $\mathbb{P}^2$, Hirzebruch surfaces $\mathbb{F}_m$, blow-ups thereof, and Enriques surfaces. In the case where the elliptic fibration is trivial then the base itself must be a Calabi–Yau two-fold, which is either a K3 surface or T^4. This is precisely the solution obtained in [55] which results in $(4,4)$ supersymmetry, and is the dual to the classic D1-D5 system [23].

At this point it is perhaps timely to recall that our description is valid away from the singular loci of τ[8]. As explained earlier, we will allow for singularities in τ, given for instance by (1.20), which have a characterisation in terms of Kodaira singular fibers. The Ricci-flatness condition then takes the form

$$K_B = -\sum_i a_i D_i\,, \tag{2.115}$$

where D_i are the Cartan divisors of the resolution of the singularity and a_i depend on the Kodaira type of the singular fiber [65, 66].

For the case of an elliptically fibered K3 surface with 24 I_1 singularities, a semi-Ricci-flat metric was constructed in [13]. The metric in the neighborhood of each I_1 fiber is given by the Ooguri-Vafa metric [67]. The semi-flat metric was constructed

[7] Of course, the one-forms σ_i can be taken to be either $\sigma_{i,L}$ or $\sigma_{i,R}$.

[8] This manifests itself *e.g.* by noting that since $c_1(B) = 2\pi\mathfrak{R}$ we would have $c_1(B)\wedge c_1(B) = 0$, contradicting the global property that

$$\int_B c_1(B)\wedge c_1(B) = 10 - h^{1,1}(B)\,.$$

by gluing the Ooguri-Vafa metric to the metric constructed in [12] around the 24 points where the fiber becomes singular. It was shown in [13] that in the limit $\mathrm{vol}(\mathbb{E}_\tau) \to 0$ the semi-flat metric reduced to a singular metric on $\mathbb{P}^1$, the base of the elliptic K3, where the singularities are exactly at the points where the fiber is singular. In [35,36] the metric in [12] was used to give some estimate of the curvature singularity, and it was argued that in the large N limit, the gravity approximation can still be trusted. One expects in higher dimensions that the metric on the base is also singular in the F-theory limit. However, as we shall discuss in section 3.2, one is still able to compute quantities of the dual CFT using this solution. It would be interesting to estimate the curvature singularities in these higher-dimensional cases, to support these findings.

In the next subsection we shall describe a supersymmetry preserving $\mathbb{Z}_M$ quotient of these solutions. This will be important for identifying the superconformal R-symmetry of the dual $(0,4)$ SCFT in the IR, and furthermore will be a key ingredient in performing the duality to 11d supergravity in section 3.3.

2.4 AdS$_3$ with 2d $\mathcal{N} = (2,2)$ and AdS$_5$ with Varying τ

Having specialised to $(0,4)$ supersymmetry we saw that the solutions are essentially unique; buoyed by this success one should ask if the solutions for $(2,2)$ supersymmetry are similarly fixed. It turns out that the answer is no and that there is a rich structure for AdS$_3$ solutions and $(2,2)$ supersymmetry, some solutions of which are given in the appendix of [60]. From the putative field theory constructions one does not expect F-theoretic solutions with non-chiral supersymmetry and this is in agreement with the holographic analysis; τ is required to be constant. Interestingly there is a second option where one retains a varying axio-dilaton. The 7d internal manifold is no longer compact and in fact combines with the AdS$_3$ factor to give AdS$_5$ solutions with varying axio-dilaton and only five-form flux. They are a natural extension of the AdS$_5$ Sasaki-Einstein solutions and have a similar geometric interpretation as we shall explain shortly. The dual field theories are 4d $\mathcal{N} = 1$ gauge theories obtained from D3 and 7-branes.

2.4.1 Torsion Conditions

Again the torsion conditions for $\mathcal{N} = (2,2)$ supersymmetry may be extracted from appendix A.1.2 by specialising the α parameters to be $\alpha_1 = -\alpha_2 = 1$ for the two spinors ξ_i. The existence of two non-vanishing Dirac Killing spinors on $\mathcal{M}_7$ implies that $\mathcal{M}_7$ supports an $SU(2)$ structure leading to a decomposition of the metric as

$$\mathcal{M}_7 = \mathcal{M}_3 \rtimes \mathcal{M}_4 \,. \tag{2.116}$$

Recalling the algebraic scalar conditions used previously it follows that the scalar bilinears A_{ij}, as defined in (A.38), satisfies

$$(\alpha_i + \alpha_j)A_{ij} = 0 \quad \Rightarrow \quad A_{11} = A_{22} = 0 \,. \tag{2.117}$$

From (2.7) we find

$$A^*_{ij}P = 0 \ , \tag{2.118}$$

and therefore for τ to vary we require $A_{12} = 0$. We may then split the cases into those with varying τ and those where τ is fixed to be constant or equivalently to the cases of vanishing A_{12} or non-trivial A_{12} respectively. Using the results of appendix A.1.2 we see that both K_{11} and K_{22} are Killing vectors. In addition one finds from (2.8) the two equations

$$i_{K_{ij}}\mathrm{d}H = -\frac{\mathrm{i}m}{2}(\alpha_i - \alpha_j)S_{ij} \ , \tag{2.119}$$

$$S_{ij}\mathrm{d}H + \frac{\mathrm{i}m}{2}(\alpha_i - \alpha_j)K_{ij} = \frac{\mathrm{e}^{-4H}}{4} i_{K_{ij}}F \ , \tag{2.120}$$

which may be used to show that the vectors K_{11} and K_{22} are also symmetries of both the warp factor and flux. They correspond to the left and right moving R-current in the putative dual SCFT respectively, as can be checked easily by computing the corresponding spinorial Lie derivatives of ξ_1 and ξ_2. In both cases the scalars S_{11} and S_{22} are constant and the spinors may be normalised such that both of these scalars are unity. This concludes the general analysis, and we must now specialise to one of the two cases. Requiring that the solution space transverse to the AdS_3 is compact implies that τ is constant. This case will be discussed in section 2.4.2, supplementary details may be found in [60]. If we relax the compactness condition then we find AdS_5 with varying τ. This allows us to classify all F-theoretic AdS_5 solutions in Type IIB with five-form flux in section 2.4.3.

2.4.2 Constant τ: AdS_3 Duals to $\mathcal{N} = (2, 2)$

In this section we provide the necessary conditions for the existence of a compact internal manifold that allows for 2d $(2, 2)$ supersymmetry, further discussion can be found in [60] where the conditions are derived and known solutions in the literature are recovered. The analysis of the torsion conditions shows that for a compact internal space constant τ is a necessary condition. In this section we consider the case where the scalar bilinear A_{12} is non-trivial. The conditions for the existence of a solution are reminiscent of the conditions found in [31] for AdS_5 solutions in

M-theory. Locally the internal metric takes the form

$$m^2\mathrm{d}s^2(\mathcal{M}_7) = (1-y\mathrm{e}^{-4H})(\mathrm{d}\psi_1+\sigma_1)^2+y\mathrm{e}^{-4H}\mathrm{d}\psi_2^2+\frac{\mathrm{e}^{-4H}}{4y(1-y\mathrm{e}^{-4H})}\mathrm{d}y^2+\mathrm{e}^{-4H}g^{(4)}(y,x)_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \tag{2.121}$$

where both ψ_1 and ψ_2 are Killing vectors and generate the expected $U(1)\times U(1)$ symmetry that is dual to the R-symmetry on the field theory side.[9] For fixed y the metric $g^{(4)}$ is Kähler with Kähler form J_4 satisfying

$$\partial_y J_4 = \frac{1}{2}\mathrm{d}_4\sigma_1\,, \qquad \partial_y \log\sqrt{g} = -\frac{4y\mathrm{e}^{-4H}}{1-y\mathrm{e}^{-4H}}\partial_y H\ , \tag{2.122}$$

where

$$\begin{aligned}\sigma_1 &= -\widehat{P}_4 + \frac{2y\mathrm{e}^{-4H}}{1-y\mathrm{e}^{-4H}}\mathrm{d}_4^c H\\ mF^{(2)} &= -\Big((\mathrm{e}^{4H}-y)\mathrm{d}\sigma_1 + 2J_4 + 4\mathrm{e}^{4H}\mathrm{d}H\wedge(\mathrm{d}\psi_1+\sigma_1)\Big)\,.\end{aligned} \tag{2.123}$$

$\widehat{P}_4$ is the Ricci-form potential for the metric $g^{(4)}$. The complex structure $J_i{}^j$ is independent of y and this allows us to rewrite (2.122) as

$$(\mathrm{d}_4\sigma_1)^+ = -\frac{4y\mathrm{e}^{-4H}}{1-y\mathrm{e}^{-4H}}\partial_y H J_4\,, \tag{2.124}$$

where $(\mathrm{d}_4\sigma_1)^+$ is the self-dual part of the two-form $\mathrm{d}_4\sigma_1$.

Examples of geometries of this type were found in [17,55] but is is likely that there exist other interesting solutions. Since the case of constant axio-dilaton is not the focus of this thesis, we leave a further analysis of these solutions and their duals for the future. We shall proceed in the next section with the analysis of non-trivially varying τ by relaxing the condition of compactness of the internal 7d manifold.

2.4.3 Varying τ: AdS$_5$ Duals to 4d $\mathcal{N}=1$

In the previous section the requirement for 2d $\mathcal{N}=(2,2)$ supersymmetry and compactness of the solution led to the result that τ is constant. We shall now relax the latter condition and find that there are non-trivial varying τ solutions which are AdS$_5$ duals to $\mathcal{N}=1$ in 4d. In the following we will provide the derivation of this starting from the present setup of AdS$_3$ solutions. We supplement this with an analysis from a direct AdS$_5\times\mathcal{M}_5^\tau$ ansatz in a later chapter 5 which shows that these are in fact all such varying τ AdS$_5$ solutions with five-form flux.

Details of the derivation are provided in appendix A.2. The solution is given in

[9] The Killing directions dual to the left and right moving R-symmetries are linear combinations of these two Killing vectors, they correspond to the diagonal and anti-diagonal.

terms of the metric

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{e}^{2H}\left(\mathrm{d}s^2(\mathrm{AdS}_3) + \frac{\mathrm{e}^{-2H}}{m^2(1-\mathrm{e}^{-2H})}\mathrm{d}H^2 + \frac{4(1-\mathrm{e}^{-2H})}{m^2}\mathrm{d}\varphi^2\right) + \frac{1}{m^2}\left[\ (\mathrm{d}\psi+\sigma)^2 + \mathrm{d}s^2(\widetilde{\mathcal{M}}_4)\right] \\
&= \mathrm{d}s^2(\mathrm{AdS}_5) + \frac{1}{m^2}\left[\ (\mathrm{d}\psi+\sigma)^2 + \mathrm{d}s^2(\widetilde{\mathcal{M}}_4)\right]\ ,
\end{aligned} \tag{2.125}$$

where $\widetilde{\mathcal{M}}_4$ is a Kähler surface. The axio-dilaton varies holomorphically over $\widetilde{\mathcal{M}}_4$ and obeys the following curvature condition

$$\mathfrak{R}_4 = 6J_4 - \mathrm{d}Q\ . \tag{2.126}$$

In particular, to find solutions we should solve this equation. Notice that for constant τ this reduces to the Kähler–Einstein condition.

As an F-theory background, this can written as

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}s^2(\mathrm{AdS}_5) + \mathrm{d}s^2(\mathcal{M}_7^\tau) \\
&= \mathrm{d}s^2(\mathrm{AdS}_5) + \frac{1}{m^2}(\mathrm{d}\psi+\sigma)^2 + \mathrm{d}s^2(\widetilde{\mathcal{M}}_4) + \frac{1}{\tau_2}(\mathrm{d}x+\tau_1\mathrm{d}y)^2 + \tau_2\mathrm{d}y^2\ ,
\end{aligned} \tag{2.127}$$

where $S^1 \hookrightarrow \mathcal{M}_7^\tau \to \mathcal{T}_6^\tau$, with the elliptically fibered three-fold $\mathbb{E}_\tau \hookrightarrow \mathcal{T}_6^\tau \to \widetilde{\mathcal{M}}_4$, which is *not Calabi-Yau*. There exists a nice reformulation of these solutions in term of an elliptically fibered Calabi–Yau four-fold. The compact part of the geometry has an obvious relation with the metrics on Sasaki–Einstein solutions and in fact may be shown to be the link of the conical base of an elliptically fibered Calabi–Yau four-fold. Specifically, that the metric

$$\mathrm{d}s^2(\mathcal{Y}_4) = \frac{1}{\tau_2}(\mathrm{d}x+\tau_1\mathrm{d}y)^2 + \tau_2\mathrm{d}y^2 + \mathrm{d}r^2 + r^2\left((\mathrm{d}\psi+\sigma)^2 + \mathrm{d}s^2(\widetilde{\mathcal{M}}_4)\right) \tag{2.128}$$

is both Ricci-flat and Kähler, where the elliptic fiber varies over the Kähler manifold $\widetilde{\mathcal{M}}_4$, see [60] for further details.

For constant τ the fibration is trivial and we reduce to the usual Sasaki–Einstein solutions, which can be written as the link of a Calabi–Yau three-fold cone. Including varying τ the solution remains Sasakian, but the Calabi–Yau condition of the 6d cone is now relaxed.

Chapter 3

The $\mathcal{N} = (0, 4)$ Holographic Dictionary

Having found the general conditions for supersymmetric solutions our task is to now solve the conditions. We then wish to establish a holographic dictionary between these solutions and 2d SCFTs. As the $(0, 4)$ solutions were essentially unique it is prudent to begin by examining these solutions in detail first. The large amount of supersymmetry leads this to be a good testing ground as computations are generally simpler.

The content of this section is taken from the bulk of [59].

3.1 Gravity analysis

Recall that the solution took the form

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}s^2(\mathrm{AdS}_3) + \mathrm{d}s^2(S^3) + \frac{1}{m_B^2}\mathrm{d}s^2(B) \ , \\ F &= -(1+*)\frac{2m}{m_B^2}J_B \wedge \mathrm{dvol}(\mathrm{AdS}_3) \ , \end{aligned} \tag{3.1}$$

with J_B the Kähler form on the base of an elliptically fibered Calabi–Yau threefold, and τ a holomorphically varying function on B, given by the complex structure of the elliptic fibration of the Calabi–Yau. This may be viewed as the near horizon of the black string solution in 6d [68].

3.1.1 Lens Space Solution

Manifest in the solution is an S^3 which has isometry group $SO(4) \simeq SU(2)_L \times SU(2)_R$, a subgroup of which realises the R-symmetry of the dual SCFT geometrically. The Killing spinors transform non-trivially under the R-symmetry but are singlets under flavour symmetries. We shall find that the Killing spinors of this solution are only charged under one of the $SU(2)$s, which identifies the R-symmetry to be the small $\mathcal{N} = (0, 4)$ superconformal R-symmetry. Furthermore, by inspection of the Killing spinors it is apparent that one can extend the solution found above by quotienting the S^3 by a discrete group $\Gamma \subset SU(2)_L$ and still preserve the same amount of supersymmetry. This generalises the solution described in section 2.3.1 to the class

$$\mathrm{AdS}_3 \times S^3/\Gamma \times B\,. \tag{3.2}$$

We will focus on the case that $\Gamma = \mathbb{Z}_M$, where the quotient has the effect of changing the period of ψ, the coordinate of the Hopf fiber, so that $\psi \sim \psi + 4\pi/M$ rather than being 4π periodic. We shall show that the Killing spinors we obtain are $SU(2)_L$ singlets, and in particular independent of ψ, therefore quotienting by $\mathbb{Z}_M$ does not break any supersymmetry.

It suffices to compute the Killing spinors in Einstein frame as this will not affect the above analysis. Moreover, as we have taken the Killing spinors to be a direct product as in (2.5) we need only consider solving the seven-dimensional Killing spinor equations (2.7)-(2.9). The Killing spinor equation obtained by restricting (2.9) to the base of the elliptically fibered Calabi–Yau is

$$\nabla_m \xi - \frac{\mathrm{i}}{2} Q_m \xi = 0\,. \tag{3.3}$$

This follows by restricting the covariantly constant Killing spinor equation of the elliptically fibered Calabi–Yau to the base by using the results for the spin connection in (2.108). Equivalently, one can notice that this is precisely the canonical spin^c Killing spinor equation on a Kähler manifold where $-Q$ is the Ricci one-form potential, as shown in the previous subsection[1]. One may take the Killing spinor on the base of an elliptically fibered Calabi–Yau manifold to be constant if one imposes suitable projection conditions. Using the relations for the spin-connection of an elliptically fibered Calabi–Yau, as computed in (2.108), one finds that the projection conditions are

$$\gamma^{34}\xi = \gamma^{56}\xi = -\mathrm{i}\xi\,, \tag{3.4}$$

where the indices are flat. In conclusion, to solve the Killing spinor equation on the base we need only to consider a constant spinor satisfying the projection conditions (3.4). Note that (2.8) is automatically satisfied thanks to (3.4) and (2.86). Moreover, holomorphicity of τ and (3.4) imply that (2.7) is also satisfied. One therefore needs only solve (2.9) for the S^3 indices.

One may use the explicit form of the flux (2.86) to reduce (2.9) on the S^3 to

$$\nabla_{\hat{a}}\xi = \frac{\mathrm{i}m}{2}\gamma_{\hat{a}}\xi\,. \tag{3.5}$$

[1]Recall that this is a local equation as Q and the metric on B are singular.

With the vielbein

$$e^1_{(S^3)} = -\frac{1}{2m}\sigma_{1,R}\,, \quad e^2_{(S^3)} = -\frac{1}{2m}\sigma_{2,R}\,, \quad e^3_{(S^3)} = -\frac{1}{2m}\sigma_{3,R}\,, \tag{3.6}$$

where $\sigma_{i,R}$ are right-invariant one-forms, one finds that the constant spinor solves this final set of conditions. The Killing spinor is therefore a constant spinor subject to the projection conditions (3.4), and therefore has four real components consistent with preserving $(0,4)$ supersymmetry. As the solution is constant in ψ, there is no ambiguity in the definition of the spinor if we quotient the S^3 by $\mathbb{Z}_M$. We may therefore replace the S^3 factor in the solution by the Lens spaces $S^3/\mathbb{Z}_M$ without breaking supersymmetry and still satisfying all equations of motion and Bianchi identities. We shall give a physical interpretation of this quotient in section 3.1.3.

Having computed the Killing spinors we may now determine the R-symmetry. On the S^3 there are six Killing vectors corresponding to the six generators of $SO(4) \simeq SU(2)_L \times SU(2)_R$. The three dual to the left-invariant one-forms are

$$\begin{aligned} k_{(1)} &= \partial_\psi\,,\ k_{(2)} = -\cos\psi\cot\theta\partial_\psi - \sin\psi\partial_\theta + \frac{\cos\psi}{\sin\theta}\partial_\varphi\,,\\ k_{(3)} &= -\sin\psi\cot\theta\partial_\psi + \cos\psi\partial_\theta + \frac{\sin\psi}{\sin\theta}\partial_\varphi\,, \end{aligned} \tag{3.7}$$

whilst the three dual to the right-invariant one-forms are

$$k_{(4)} = \frac{\sin\varphi}{\sin\theta}\partial_\psi + \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi\,,\ k_{(5)} = -\frac{\cos\varphi}{\sin\theta}\partial_\psi + \sin\varphi\partial_\theta + \cot\theta\cos\varphi\partial_\varphi\,,\ k_{(6)} = \partial_\varphi\,, \tag{3.8}$$

with each set satisfying the $SU(2)$ Lie algebra. The spinorial Lie-derivative (also known as Kosmannn spinorial Lie derivative) along a Killing direction, K, is defined to be

$$\mathcal{L}_K\epsilon = \left(K^\mu\nabla_\mu + \frac{1}{8}(\mathrm{d}K)_{\nu_1\nu_2}\gamma^{\nu_1\nu_2}\right)\epsilon\,. \tag{3.9}$$

In order to ascertain along which directions the Killing spinor is charged one computes the spinorial Lie derivative along these directions. We find that the Killing spinor is invariant under the left-invariant Killing vectors and charged under the right-invariant Killing vectors. This implies that we can take the quotient by $\Gamma \subset SU(2)_L$, preserving the same amount of supersymmetry. Moreover, as discussed above this means that we can identify $SU(2)_R$ with the $SU(2)_r$ R-symmetry of the dual SCFT. We note that the spinorial Lie derivative is frame independent (subject to preserving the same orientation, which is correlated with the choice of $SU(2)$ under which the Killing spinors are charged) and therefore this result is non-ambiguous.

It is a well known fact in the literature that performing a T-duality along a Killing direction with vanishing spinorial Lie-derivative for the Killing spinor along the Killing vector leads to a Killing spinor in the dual solution. It is clear from the results above that one may dualize along the Hopf fiber without breaking supersymmetry. This will be used later on to determine the dual M-theory solution.

3.1.2 Flux Quantisation

To complete the solution, we need to ensure that the five-form field strength, F, is properly quantized through all the integral five-cycles in the 7d manifold transverse to AdS_3. We impose that

$$n(M_\alpha) = \frac{1}{(2\pi l_s)^4 g_\mathrm{s}} \int_{M_\alpha} F \in \mathbb{Z} \tag{3.10}$$

for all $M_\alpha \in H_5(\mathcal{M}_7, \mathbb{Z})$. The five cycles which contribute are of the form $S^3 \times \mathcal{C}$, where $\mathcal{C}$ is any two-cycle in the base B of the Calabi–Yau. We therefore find [2]

$$\begin{aligned} n(M_\alpha) &= -\frac{\mathrm{vol}(S^3/\mathbb{Z}_M)}{(2\pi l_s)^4 g_\mathrm{s} 4m^2 m_B^2} \int_{Y_3} \omega_0 \wedge \pi^* J_B \wedge \omega_\alpha &(3.11)\\ &= -\frac{\mathrm{vol}(S^3/\mathbb{Z}_M)}{(2\pi l_s)^4 g_\mathrm{s} 4m^2 m_B^2} \int_{C_\alpha} J_B \,, &(3.12)\end{aligned}$$

where the C_α form a basis of cycles in $H_2(B, \mathbb{Z})$.

The possible bases B for an elliptic Calabi–Yau threefold, as listed earlier, are projective, and therefore also Hodge manifolds [69], and moreover they admit an integral Kähler form. As J_B is dual to a curve, we in fact have that B is not only a Hodge manifold, but we in fact pick the Hodge metric on it. This implies that we can take

$$\int_{C_\alpha} J_B = k_\alpha \;\in\; \mathbb{Z}^+ \,. \tag{3.13}$$

Using (3.13) we find that $n(M_\alpha)$ are integer if we impose

$$N = \frac{\mathrm{vol}(S^3/\mathbb{Z}_M)}{4m^2 m_B^2 (2\pi l_s)^4 g_\mathrm{s}} \in \mathbb{Z} \,. \tag{3.14}$$

3.1.3 Brane Solutions and the Interpretation of the Quotient

In this subsection we shall give an interpretation of the $\mathbb{Z}_M$ quotient performed in section 3.1.1. To do so we shall construct smeared brane solutions whose near-horizon geometry is

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathrm{AdS}_3) + \frac{1}{4m^2}\mathrm{d}s^2(S^3/\mathbb{Z}_M) + \frac{1}{m_B^2}\mathrm{d}s^2(B) \,. \tag{3.15}$$

[2] We have defined $\mathrm{dvol}(S^3/\mathbb{Z}_M) = \sigma_{1,R} \wedge \sigma_{2,R} \wedge \sigma_{3,R}$ which gives $\mathrm{vol}(S^3/\mathbb{Z}_M) = \frac{16\pi^2}{M}$. Notice that this is not the volume form of the unit radius Lens space $S^3/\mathbb{Z}_M$.

We shall need to combine various D3-brane solutions, employing the harmonic function rule (see [70] for a review).

We shall use this strategy to obtain a UV completion of the AdS_3 solution that we have in Type IIB in the near-horizon limit, which we refer to as the "pre near-horizon limit". In fact, as we will show below, we can construct *two distinct* such solutions, both flowing to the same near-horizon geometry. We wish to consider N D3-branes wrapping $\mathbb{R}^{1,1} \times C$ where C is the curve in the base of Y_3, Poincaré dual to the Kähler form of the base. We shall first consider a solution in the background of M KK-monopoles and later in the background $\mathbb{R}^4$. To realise the D3-branes extended along the curve Poincaré dual to $J = e^{12} + e^{34}$, with $\mathrm{d}s^2(B) = e_1^2 + \cdots + e_4^2$ we shall formally view this as two stacks of D3-branes [71]. The first stack will extend along $\mathbb{R}^{1,2} \times C_{12}$, where C_{12} is the curve dual to e^{12}, and the second stack along $\mathbb{R}^{1,2} \times C_{34}$ each with the same number of branes, N.

We begin by briefly recalling the metric for M KK-monopoles and give a few comments that will be useful for later discussion. The metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_5^2 + \mathrm{d}s^2_{TN_M}\,, \tag{3.16}$$

where $\mathrm{d}s^2_{TN_M}$ is the Taub-NUT metric[3]

$$m_B^2 \mathrm{d}s^2_{TN_M} = \left(1 + \frac{M}{r}\right)\left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)\right) + \left(1 + \frac{M}{r}\right)^{-1} (\mathrm{d}\psi + M\cos\theta \mathrm{d}\varphi)^2\,. \tag{3.17}$$

This metric is well-known to be hyper-Kähler and hence Ricci-flat.

This metric approaches the singular (for $M > 1$) metric on $\mathbb{R}^4/\mathbb{Z}_M$ as $r \to 0$, whilst asymptotically, as $r \to \infty$, it approaches the cylinder $\mathbb{R}^3 \times S^1$. One can set $M = 0$ in the metric, obtaining exactly the flat metric on $\mathbb{R}^3 \times S^1$. Moreover, choosing as harmonic function $\mathcal{H}(r) = 1 + \frac{M}{r} \to \frac{M}{r}$, a simple change of coordinates shows that this is exactly the metric on $\mathbb{R}^4/\mathbb{Z}_M$. This can be interpreted as saying that in the "near-horizon" limit the Taub-NUT metric approaches the latter.

Let us first write the Type IIB solution corresponding to N D3-branes wrapping $\mathbb{R}^{1,2} \times C_{12}$,

$$\begin{aligned} m_B^2 \mathrm{d}s^2(D3, 12) &= \mathcal{H}^{-1/2}(r)(-\mathrm{d}t^2 + \mathrm{d}x^2 + e_1^2 + e_2^2) + \mathcal{H}^{1/2}(r)(\mathrm{d}s^2_{TN} + e_3^2 + e_4^2)\,, \\ C^{(4)} &= \frac{1}{m_B^2}\left(1 - \frac{1}{\mathcal{H}(r)}\right) \mathrm{d}t \wedge \mathrm{d}x \wedge e^{12}\,. \end{aligned} \tag{3.18}$$

[3]Strictly speaking, the Taub-NUT metric has $M = 1$ and this is non singular near to $r \to 0$. The metric with $M > 1$ has an $\mathbb{R}^4/\mathbb{Z}_M$ singularity in the interior, and this can be resolved by replacing the single centre metric with a Gibbons-Hawking multi-centre metric, where near to each centre the metric looks like $\mathbb{R}^4$. This metric develops $M - 1$ two-cycles, that collapse to zero size in the single centre singular metric.

To wrap $\mathbb{R}^{1,2} \times C_{34}$ we simply relabel $12 \leftrightarrow 34$. We have inserted the D3-branes into the background of M KK-monopoles. In particular, as remarked above, we shall *smear* the D3-branes completely along the 34 directions in the manifold B, this has the affect of making the function $\mathcal{H}(r)$ harmonic on Taub-NUT and not the overall transverse space to the stack of D3s. If we now use the harmonic function rule on these two configurations we obtain the solution

$$\begin{aligned} m_B^2 \mathrm{d}s^2 &= \mathcal{H}(r)^{-1}(-\mathrm{d}t^2 + \mathrm{d}x^2) + \mathcal{H}(r)\mathrm{d}s^2_{TN} + \mathrm{d}s^2(B) \\ C^{(4)} &= \frac{1}{m_B^2}\left(1 - \frac{1}{\mathcal{H}(r)}\right)\mathrm{d}t \wedge \mathrm{d}x \wedge J_B \end{aligned} \tag{3.19}$$

As commented above H must be harmonic on Taub-NUT, as such we may take

$$\mathcal{H}(r) = 1 + \frac{q_N}{r}\,, \qquad \text{with} \qquad q_N \equiv \frac{(2\pi\ell_s)^4 N m_B^4}{16\pi^2}\,, \tag{3.20}$$

and N the number of D3-branes. The metric takes the form

$$m_B^2 \mathrm{d}s^2 = \frac{r}{r+q_N}(-\mathrm{d}t^2 + \mathrm{d}x^2) + \frac{r+q_N}{r}\mathrm{d}s^2_{TN} + \mathrm{d}s^2(B)\,. \tag{3.21}$$

We recall that B is the base of an elliptically fibered Calabi–Yau threefold and as such this necessarily requires τ to vary in the solution. This is an Einstein-frame solution to Type IIB supergravity with D3-branes and varying τ.

Let us now take the near-horizon limit, $r \to 0$. We have

$$\begin{aligned} m_B^2 \mathrm{d}s^2_{r\to 0} &= \frac{r}{q_N}(-\mathrm{d}t^2 + \mathrm{d}x^2) + \mathrm{d}s^2(B) + \frac{q_N M}{r^2}\left(\mathrm{d}r^2 + r^2 \mathrm{d}s^2(S^2)\right) + \frac{q_N}{M}(\mathrm{d}\psi + M\cos\theta \mathrm{d}\phi)^2 \\ &= \frac{r}{q_N}(-\mathrm{d}t^2 + \mathrm{d}x^2) + q_N M\frac{\mathrm{d}r^2}{r^2} + \mathrm{d}s^2(B) + q_N M(\mathrm{d}s^2(S^3/\mathbb{Z}_M))\,. \end{aligned} \tag{3.22}$$

If we make the redefinition $Mq_N = m_B^2(4m^2)^{-1}$ and the change of coordinate $r = 4q_N^2 M\rho^2$ we obtain

$$\mathrm{d}s^2 = \frac{1}{m^2}\left(\rho^2(-\mathrm{d}t^2 + \mathrm{d}x^2) + \frac{\mathrm{d}\rho^2}{\rho^2}\right) + \frac{1}{4m^2}\mathrm{d}s^2(S^3/\mathbb{Z}_M) + \frac{1}{m_B^2}\mathrm{d}s^2(B)\,, \tag{3.23}$$

whilst the five-form becomes

$$F = (1+*)\frac{2m}{m_B^2}\mathrm{dvol}(\mathrm{AdS}_3) \wedge J_B\,, \tag{3.24}$$

which recovers exactly the AdS_3 solution. We have done this by inserting M KK-monopoles into the background of N D3-branes wrapping a curve, C dual to J_B, on the base of an elliptically fibered Calabi–Yau threefold.

Let us now consider a different pre near-horizon limit of the AdS_3 solution. This will be obtained by replacing the Taub-NUT metric in the Type IIB solution by the flat space quotient $\mathbb{R}^4/\mathbb{Z}_M$. We shall see that the near-horizon solution agrees with the Taub-NUT solution. We may use the previous results to immediately write down the metric[4]

$$\begin{aligned}m_B^2\mathrm{d}s^2 =&\mathcal{H}_2(R)^{-1}(-\mathrm{d}t^2+\mathrm{d}x^2)+\mathcal{H}_2(R)\left(\mathrm{d}R^2+\frac{R^2}{4}((\mathrm{d}\psi+\cos\theta\mathrm{d}\varphi)^2+\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2)\right)\\ &+\mathrm{d}s^2(B)\end{aligned} \tag{3.25}$$

where now $\mathcal{H}_2(R)$ is a harmonic function on $\mathbb{R}^4$ and we take

$$\mathcal{H}_2(R)=1+\frac{\widetilde{q_N}}{R^2}\qquad\text{with}\qquad\widetilde{q_N}\equiv\frac{(2\pi\ell_\mathrm{s})^4MNm_B^4}{4\pi^2}\,. \tag{3.26}$$

This harmonic function should be contrasted with (3.20) in the Taub-NUT case. The self-dual five-form flux takes the form

$$F=(1+*)\frac{1}{m_B^4}\mathrm{d}\mathcal{H}_2(R)^{-1}\wedge\mathrm{d}t\wedge\mathrm{d}x\wedge J_B\,. \tag{3.27}$$

Taking the near-horizon limit, $R\to 0$ we obtain

$$m_B^2\mathrm{d}s^2=\frac{R^2}{\widetilde{q_N}}(-\mathrm{d}t^2+\mathrm{d}x^2)+\frac{\widetilde{q_N}}{R^2}\mathrm{d}R^2+\frac{\widetilde{q_N}}{4}\mathrm{d}s^2(S^3/\mathbb{Z}_M)+\mathrm{d}s^2(B)\,. \tag{3.28}$$

After rescaling R as $R\to\widetilde{q_N}R$ and identifying the inverse radius of AdS_3 to be $\widetilde{q_N}=\frac{m_B^2}{m^2}$ one recovers precisely the $\mathrm{AdS}_3\times S^3/\mathbb{Z}_M$ solution

$$m_B^2\mathrm{d}s^2=\frac{R^2}{m^2}(-\mathrm{d}t^2+\mathrm{d}x^2)+\frac{1}{m^2R^2}\mathrm{d}R^2+\frac{1}{4m^2}\mathrm{d}s^2(S^3/\mathbb{Z}_M)+\mathrm{d}s^2(B) \tag{3.29}$$

in perfect agreement with (2.112). The flux becomes

$$F=(1+*)\frac{2m}{m_B^2}\mathrm{dvol}(\mathrm{AdS}_3)\wedge J_B\,, \tag{3.30}$$

in agreement with (2.113).

We have constructed two different UV completions of the Type IIB AdS_3 solution that is our main interest. To do so, we needed to make some technical simplifications, regarding smearing of the branes and the application of the harmonic sum rule. The resulting solutions are therefore not the fully localized brane solutions, before taking the near-horizon limit, which are typically very difficult to construct. However these solutions will still be useful in our discussion. Moreover, we should also keep in mind that the metric on B and τ were singular in the near-horizon limit and this feature will remain.

[4]Of course here we can simply take ψ to have period $4\pi/M$.

Notice that for any N and any M, asymptotically the metric (3.21) goes to $\mathbb{R}^{1,1} \times \mathbb{R}^3 \times S^1 \times B$. This is the metric far away from the N D3-branes. On the other hand, the metric (3.25) asymptotically goes to $\mathbb{R}^{1,1} \times \mathbb{R}^4/\mathbb{Z}_M \times B$. So these are clearly two different UV completions of the near-horizon geometry. This becomes particularly instructive in the case of $M = 1$: in this case both asymptotic spaces are smooth, however the solution in the presence of 1 KK-monopole comprises an asymptotic $\mathbb{R}^3 \times S^1$ geometry, whilst the solution with *no* KK-monopoles comprises an asymptotic $\mathbb{R}^4$ geometry. However, they flow to exactly the same $\mathrm{AdS}_3 \times S^3$ solution in the IR.

The interpretation of this fact is that in the IR the field theories constructed from the two different UV setups, flow to the "same" SCFT in the large N limit. This means that in this limit for example the two theories must have the same central charges, in the large N limit. However, sub-leading corrections to the central charges may be possible.

Notice that one may set $M = 0$ without any immediate problem in (3.21), obtaining the metric

$$m_B^2 \mathrm{d}s^2 = \frac{r}{r+q_N}(-\mathrm{d}t^2+\mathrm{d}x^2)+\mathrm{d}s^2(B)+\frac{r+q_N}{r}\left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) + \mathrm{d}\psi^2\right) . \tag{3.31}$$

Notice that the Calabi–Yau base is a direct product with the remaining six-dimensional metric. Computing the curvature invariants of the six-dimensional metric, we find

$$\begin{aligned} \mathrm{R} &= 0\,, \\ \mathrm{R}_{AB}\mathrm{R}^{AB} &= \frac{3q_N^4}{2r^2(q_N+r)^6} \\ \mathrm{R}_{ABCD}\mathrm{R}^{ABCD} &= \frac{q_N^2(11q_N^2+32q_N r+48r^2)}{2r^2(q_N+r)^6} \end{aligned} \tag{3.32}$$

and therefore the metric is singular at $r = 0$. In fact, upon taking the near-horizon limit there is no longer an AdS_3 factor. In other words, putting the D3-branes transverse to the space $\mathbb{R}^3 \times S^1$ gives rise to a solution that does not contain an AdS_3 factor in the IR, and in fact has a curvature singularity as $r \to 0$.

3.2 F-theory Holographic Central Charges

In this section we compute the central charges for the solution derived in section 2.3.1. As was noted previously, the metric on the base B, which is induced from the Calabi–Yau metric is singular. We shall circumvent potential problems arising with singular metrics, by carrying out our computations in the smooth Calabi–Yau threefold.

3.2.1 Leading Order Central Charges

The leading order term for the central charges is given by the Brown-Henneaux formula [72] as summarised in appendix B.1.1. Evaluating (B.6) for the solution we find the leading order central charges to be

$$
\begin{aligned}
(c_L^{\mathrm{IIB}})^{(2)} = (c_R^{\mathrm{IIB}})^{(2)} = c_{\mathrm{sugra}}^{\mathrm{IIB}} = \frac{3\mathrm{vol}(M_7)}{2mG_N^{(10)}} &= N^2 \frac{3\mathrm{vol}(S^3/\mathbb{Z}_M)\mathrm{vol}(B)32\pi^2}{\mathrm{vol}(S^3/\mathbb{Z}_M)^2} \\
&= 6N^2 M \mathrm{vol}(B)\,.
\end{aligned} \tag{3.33}
$$

We denote by $c^{(a)}$ the $\mathcal{O}(N^a)$ contribution to the central charge.

In a smooth geometry we would compute the volume of the base B using the metric. However, as we emphasised repeatedly, the metric of this space is singular. There is a smooth Ricci-flat metric on the putative elliptically fibered Calabi–Yau Y_3. The way we will work around the absence of a smooth metric on B is to compute the volume in the elliptic Calabi–Yau as follows. The $(1,1)$-form dual to B is ω_0, and the volume of the divisor can be computed by

$$
\mathrm{vol}(B) = \frac{1}{2}\int_{Y_3} \omega_0 \wedge \pi^* J_B \wedge \pi^* J_B = \frac{1}{2}\int_B J_B \wedge J_B\,. \tag{3.34}
$$

Furthermore the latter integral can be evaluated by first using the fact that the curve wrapped by the D3-branes, C, is Poincaré dual to the Kähler form J_B and then using intersection theory to write

$$
\mathrm{vol}(B) = \frac{1}{2}\int_C J = \frac{1}{2} C \cdot C\,. \tag{3.35}
$$

Using this identification we can rewrite the central charge in terms of the self-intersection of the curve C in B as

$$
(c_L^{\mathrm{IIB}})^{(2)} = (c_R^{\mathrm{IIB}})^{(2)} = c_{\mathrm{sugra}}^{\mathrm{IIB}} = 3N^2 M\, C\cdot C\,. \tag{3.36}
$$

Since $\mathrm{vol}(B) > 0$ the curve wrapped by the D3-branes must have positive self-intersection in B. Using the adjunction formula (see the footnote)[5] one can express the constraint $C \cdot C > 0$ as

$$
C\cdot C = 2(g-1) + c_1(B)\cdot C > 0\,. \tag{3.37}
$$

[5]Consider a curve $C \subset B$, the adjunction formula reads

$$
K_C = (K_B + C)|_C\,,
$$

where K_C and K_B are the canonical classes of C and B, respectively. For a genus g curve this implies

$$
2(g-1) = C\cdot C - c_1(B)\cdot C\,.
$$

At this point we should comment about the relation of our setup to the strings in minimal 6d SCFTs, also known as non-Higgsable clusters (NHCs) [73], whose central charges were computed in [54]. The geometric condition for the NHCs is that the base of the Calabi–Yau threefold is locally $\mathcal{O}(-n) \to \mathbb{P}^1$. The curve that is wrapped by the D3-brane is the base $C^{\mathrm{NHC}} = \mathbb{P}^1$, which has self-intersection

$$C^{\mathrm{NHC}} \cdot C^{\mathrm{NHC}} = -n < 0\,, \qquad n = 3, 4, 6, 8, 12\,, \tag{3.38}$$

and can be collapsed. This singular limit corresponds to the conformal point. In appendix B.2.3 the geometry of these NHCs is briefly discussed. The negative self-intersection implies that C^{NHC} is not ample, and consequently that these 2d NHC strings do not directly fit into the framework discussed in this thesis.

3.2.2 $c_L^{\mathbf{IIB}} - c_R^{\mathbf{IIB}}$ at Sub-leading Order from Anomaly Inflow

The sub-leading contribution is obtained using anomaly inflow [37]. The difference of the left and right central charges appears as the coefficient in front of the gravitational Chern-Simons term in the bulk action [74]

$$S_{CS}(\Gamma_{\mathrm{AdS}_3}) = \frac{c_L^{\mathrm{IIB}} - c_R^{\mathrm{IIB}}}{96\pi} \int_{\mathrm{AdS}_3} \omega_{CS}(\Gamma_{\mathrm{AdS}_3})\,. \tag{3.39}$$

To determine this coefficient we consider the three dimensional terms which arise from the dimensional reduction of the Chern-Simons terms in the worldvolume action of 7-branes. The Chern-Simons terms for a D7-brane were computed in [75] and are given in terms of the curvature two-forms of the tangent and normal bundles of the brane worldvolume, $\mathcal{R}_T$ and $\mathcal{R}_N$, respectively,

$$\mu_7 \int_{\mathcal{W}_8} C^{(4)} \wedge \sqrt{\frac{\hat{A}(4\pi^2 {\ell_{\mathrm{s}}}^2 \mathcal{R}_T)}{\hat{A}(4\pi^2 {\ell_{\mathrm{s}}}^2 \mathcal{R}_N)}}\, \mathrm{Tr}\left(\mathrm{e}^{2\pi {\ell_{\mathrm{s}}}^2 \mathcal{F}}\right) \subset S_{\mathrm{D7}}\,, \tag{3.40}$$

where

$$\mu_7 = \frac{1}{(2\pi)^7 {\ell_{\mathrm{s}}}^8}\,, \tag{3.41}$$

is the charge of a single D7-brane, C_4 is the potential of the five-form flux and $\mathcal{F}$ is the gauge invariant field strength of the gauge fields on the D7-brane. The trace is performed in the fundamental representation of the gauge group. For the computation of the $\mathcal{O}(N)$ corrections to the central charges we will only be interested in the terms coming from the tangent bundle of the D7-brane. Thus below we simply write $\mathcal{R} \equiv \mathcal{R}_T$. Up to the required order the A-roof genus $\hat{A}$ is given by

$$\hat{A}(\mathcal{R}) \;\; = \;\; 1 + \frac{1}{12(4\pi)^2} \mathrm{Tr}(\mathcal{R} \wedge \mathcal{R})\,. \tag{3.42}$$

As we consider only I_1 singularities our set-up consists of *single* 7-branes wrapped on curves C_x in the base[6]. Note that not all of these 7-branes can be transformed into D7-branes under an $SL(2,\mathbb{Z})$ transformation. Imposing that the elliptic fibration is Calabi–Yau results in the constraint

$$[\Delta] = 12c_1(B) = \sum_x \omega_x\,, \tag{3.43}$$

where ω_x are the two-forms dual to the curves C_x wrapped by the 7-branes.

Consider a single D7-brane whose world-volume extends along $\mathcal{W}_8 = \mathrm{AdS}_3 \times S^3/\mathbb{Z}_M \times C_x$. From the D7-brane Wess-Zumino term we obtain the 3d Chern-Simons term

$$\begin{aligned} S_{CS}(\Gamma_{\mathrm{AdS}_3}) &= \frac{\mu_7 \pi^2 {\ell_\mathrm{s}}^4}{24 g_\mathrm{s}} \int_{\mathcal{W}_8} C^{(4)} \wedge \mathrm{Tr}(\mathcal{R}\wedge\mathcal{R}) \\ &= -\frac{\mu_7 \pi^2 {\ell_\mathrm{s}}^4}{24 g_\mathrm{s}} \int_{\mathcal{W}_8} F \wedge \omega_{CS} \\ &= \frac{\mu_7 \pi^2 {\ell_\mathrm{s}}^4 \mathrm{vol}(S^3/\mathbb{Z}_M)}{24 g_\mathrm{s}(2m)^2 m_B^2} \int_{C_x} J_B \int_{\mathrm{AdS}_3} \omega_{CS}(\Gamma_{\mathrm{AdS}_3}) \\ &= \frac{N}{192\pi} \int_B J_B \wedge \omega_x \int_{\mathrm{AdS}_3} \omega_{CS}(\Gamma_{\mathrm{AdS}_3})\,, \end{aligned} \tag{3.44}$$

where we have used the fact the trace over the fundamental representation of the gauge group is 1 as only one D7-brane is wrapped on C_x.

As $C^{(4)}$ is invariant under $SL(2,\mathbb{Z})$ transformations, each 7-brane gives rise to the same contribution to the 3d Chern-Simons term [37]. To obtain the total contribution we therefore sum the terms arising from each 7-brane

$$S_{CS}(\Gamma_{\mathrm{AdS}_3}) = \frac{N}{192\pi} \sum_x \int_B J_B \wedge \omega_x \int_{\mathrm{AdS}_3} \omega_{CS}(\Gamma_{\mathrm{AdS}_3}) \tag{3.45}$$

$$= \frac{N}{16\pi} \int_B J_B \wedge c_1(B) \int_{\mathrm{AdS}_3} \omega_{CS}(\Gamma_{\mathrm{AdS}_3})\,. \tag{3.46}$$

We evaluate the integral over the base by pulling back to the smooth Calabi–Yau

$$\int_{Y_3} \omega_0 \wedge \pi^* J_B \wedge \pi^* c_1(B) = \int_B J_B \wedge c_1(B) = c_1(B)\cdot C\,. \tag{3.47}$$

Using this relation we determine from the coefficient of (3.45) the difference of the left and right central charges to be

$$(c_L^{\mathrm{IIB}})^{(1)} - (c_R^{\mathrm{IIB}})^{(1)} = 6N c_1(B)\cdot C\,. \tag{3.48}$$

[6] This can be easily generalised to other 7-brane singularities, by including suitable normalisations to the trace appearing in (3.40).

3.2.3 Level of the Superconformal R-symmetry

In this section we compute the level k_r of the superconformal R-symmetry. The relation $c_R = 6k_r$ and (3.36) imply that the leading order contribution to the level is given by

$$k_r^{(2)} = \frac{1}{2} N^2 M C \cdot C \,. \tag{3.49}$$

To compute the sub-leading order term we restrict to the case of $M = 1$[7] and proceed by gauging the $SO(4)_T$ isometry of the S^3 in the supergravity solution. The procedure for computing the level follows [76, 77], where one first deforms the metric on the S^3 to contain connections, which depend on AdS_3

$$\mathrm{d}s^2_{S^3} \to (\mathrm{d}x^p - A^{pq}x^q)(\mathrm{d}x^p - A^{pr}x^r) \,, \tag{3.50}$$

where $\sum_{p=1}^4 (x^p)^2 = 1$. These connections $A^{pq} = -A^{qp}$ are one-forms on AdS_3 and are identified with the $SO(4)_T \simeq SU(2)_L \times SU(2)_R$ gauge fields for the superconformal R-symmetry $SU(2)_R$ and the flavour symmetry $SU(2)_L$. The deformed five-form flux is [76]

$$F_5'|_{M=1} = -\frac{4\pi^2}{m^2 m_B^2}(1+*)\left((e_3 - \chi_3) \wedge J_B\right) \,, \tag{3.51}$$

where e_3 is the volume form on the sphere bundle satisfying $\int_{S^3} e_3 = 1$ and $\mathrm{d}e_3 = \chi_4$, χ_4 being the Euler class of the sphere bundle. The additional term χ_3, a three-form on AdS_3 satisfying $\mathrm{d}\chi_3 = \chi_4$, is required for $\mathrm{d}F_5'|_{M=1} = 0$.

The reduction of the Chern-Simons term for D7-branes wrapped on this deformed metric gives rise to Chern-Simons terms for the $SO(4)_T$ gauge fields. Upon inserting the deformed flux (3.51) into the D7-brane Chern-Simons term and summing over all 7-branes as above one finds, in addition to the gravitational Chern-Simons term,

$$S_{CS}(A_T)|_{M=1} = \frac{N}{8\pi} c_1(B) \cdot C \int_{\mathrm{AdS}_3} \left(\omega_{CS}(A_R) + \omega_{CS}(A_L)\right) \,, \tag{3.52}$$

where the additional factor of 2 arises from expressing the trace over the fundamental representation of $SU(2)_R$ and $SU(2)_L$ instead of the vector representation of $SO(4)_T$. The level of the superconformal $SU(2)_r$ R-symmetry can be extracted from the coefficient of Chern-Simons term after multiplication by 4π, namely

$$S_{CS}(A) = \frac{k_r}{4\pi} \int_{\mathrm{AdS}_3} \omega_{CS}(A) \,, \tag{3.53}$$

where $A = \mathrm{i}A^a \sigma_a/2$. From the coefficient of $\omega_{CS}(A_R)$ the sub-leading order term in

[7] In [68] the central charge for $M > 1$ is obtained from 5d gauged supergravity, and they find agreement in the $M = 1$ limit with the result presented here. We shall make further comments about their work in the later M-theory section 3.4.

the level of the superconformal R-symmetry can be extracted and found to be

$$k_r^{(1)}|_{M=1} = \frac{1}{2} N c_1(B) \cdot C \,. \tag{3.54}$$

For the cases with $M > 1$, the isometry group of the solution is broken to $SU(2)_R \times U(1)_L$. Naively, to compute the level of the superconformal R-symmetry one should still gauge the $SU(2)_R$ by introducing gauge fields for this isometry, analogous to the $M = 1$ case. Formally, this gives exactly the same result as (3.54); however this is not the complete contribution, as one would have to take into account the effects of the M KK-monopoles. As we shall see in section 3.4.3, on the 11d supergravity side this will be captured by gauging the $SU(2)_{\text{11d}}$ isometry of an S^2, which arises from the base of the $S^3/\mathbb{Z}_M$ Hopf fibration. However, it should be noted that $SU(2)_R$ is different from $SU(2)_{\text{11d}}$, and one can check explicitly that in fact the latter is *not* an isometry of the Type IIB solution.

3.2.4 Summary: Central Charges from F-theory

From the computations carried out in this section the central charges in Type IIB supergravity for $M = 1$ are given by

$$c_R^{\text{IIB}}|_{M=1} = 3N^2 \, C \cdot C + 3N c_1(B) \cdot C \,, \tag{3.55}$$

$$c_L^{\text{IIB}}|_{M=1} = 3N^2 \, C \cdot C + 9N c_1(B) \cdot C \,. \tag{3.56}$$

In this section we have only computed these central charges to sub-leading order in N. We expect $\mathcal{O}(1)$ corrections to arise from one loop computations and will comment on these in section 3.5.4, where we compare the central charges computed via anomaly inflow and supergravity solutions. We further point out that the superconformal algebra mandates that the right-moving central charge belongs to $6\mathbb{Z}$. To see this explicitly we make use of the adjunction formula (3.37) and rewrite it as

$$c_R^{\text{IIB}}|_{M=1} = 6N^2(g-1) + 3N(N+1) c_1(B) \cdot C \,, \tag{3.57}$$

which exhibits manifestly that the expression is a multiple of six, generalising to any N the property of the $N = 1$ right central charge, observed in (1.27).

For $M > 1$ we obtain

$$c_R^{\text{IIB}}|_{M>1} = 3MN^2 \, C \cdot C + \delta c_R^{IIB} \tag{3.58}$$

$$c_L^{\text{IIB}}|_{M>1} = 3MN^2 \, C \cdot C + \delta c_L^{IIB} + 6N c_1(B) \cdot C \,. \tag{3.59}$$

As explained in the previous section, the computation of the level of the superconformal R-symmetry for $M > 1$ is troublesome. Instead, we uplift our Type IIB solution to 11d supergravity in the next section. In doing so we will be able to compute the

$\mathcal{O}(N)$ contributions to the $M > 1$ central charges, as well as $\mathcal{O}(1)$ corrections.

3.3 M/F-Duality and AdS$_3$ Solutions in M-theory

The solution found above in Type IIB supergravity is singular at the loci above which τ degenerates. We circumnavigated this problem by computing the central charges of the solutions in terms of the volume of the base B in the smooth Calabi–Yau, where it is well-defined. To substantiate this we can utilize M/F-duality: by T-dualizing and uplifting to M-theory, the elliptically fibered Calabi–Yau threefold becomes manifest in the geometry[8]. Assuming that there are only I_1 fibers, the elliptic Calabi–Yau threefold is smooth, as can be seen by direct computation. There exists a smooth Ricci-flat metric on this space by Yau's theorem [78] and we may use this metric to compute the central charge.

3.3.1 Dual 11d Supergravity Solution

In this subsection we shall perform a T-duality along the Hopf fiber of the $S^3/\mathbb{Z}_M$ to Type IIA and then perform the uplift to 11d supergravity. As noted in section 3.1.1 this will preserve all supersymmetries of the original solution.

Recall that the Type IIB solution in string frame takes the form

$$\mathrm{d}s^2(\mathcal{M}_{IIB}) = \frac{1}{\sqrt{\tau_2}}\left[\mathrm{d}s^2(\mathrm{AdS}_3) + \frac{1}{m_B^2}\mathrm{d}s^2(B) + \frac{1}{4m^2}(\sigma_{1,L}^2 + \sigma_{2,L}^2 + \sigma_{3,L}^2)\right] , \quad (3.60)$$

$$F = -\frac{2m}{m_B^2} J_B \wedge \mathrm{dvol}(\mathrm{AdS}_3) - \frac{1}{4m^2 m_B^2} J_B \wedge \sigma_{1,L} \wedge \sigma_{2,L} \wedge \sigma_{3,L} \, , \quad (3.61)$$

$$\tau = \tau_1 + \mathrm{i}\tau_2 = C^{(0)} + \mathrm{i}e^{-\phi} \, . \quad (3.62)$$

The metric defined by $\frac{1}{4}(\sigma_{1,L}^2 + \sigma_{2,L}^2 + \sigma_{3,L}^2)$ is that of the round, unit radius Lens space, $S^3/\mathbb{Z}_M$. This is obtained by quotienting the Hopf fiber, $\sigma_{3,L}$ in our conventions, by the discrete group $\mathbb{Z}_M$ which has the effect of reducing the period of ψ from 4π to $4\pi/M$. Recall that M corresponds to the number of KK-monopoles in the solution before going to the near-horizon limit, as was discussed in section 3.1.3.

Before performing the T-duality along the Killing vector ∂_ψ we shall absorb the factor of $4m^2$ into the definition of ψ by making the change of coordinates

$$y = \frac{m_B}{2m}\psi \, , \quad (3.63)$$

where y now has period $\frac{2\pi m_B}{Mm}$. If we now perform the T-duality along ∂_y we obtain

[8] Note that we will not perform the duality chain advocated in section 2.2.6 here, instead using a more conventional T-duality along the Hopf fibration of the quotiented S^3.

the Type IIA solution

$$\mathrm{d}s^2(\mathcal{M}_{IIA}) = \frac{1}{\sqrt{\tau_2}}\left[\mathrm{d}s^2(\mathrm{AdS}_3) + \frac{1}{m_B^2}\mathrm{d}s^2(B) + \frac{1}{4m^2}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)\right] + \frac{\sqrt{\tau_2}}{m_B^2}\mathrm{d}y^2\,, \tag{3.64}$$

$$C^{(1)} = \tau_1 \mathrm{d}y\,, \tag{3.65}$$

$$\mathrm{e}^{-\hat{\phi}} = \tau_2^{3/4}\,, \tag{3.66}$$

$$B^{IIA} = -\frac{\cos\theta}{2mm_B}\mathrm{d}y\wedge\mathrm{d}\varphi\,, \tag{3.67}$$

$$F_4^{IIA} = \frac{1}{2mm_B^2}\mathrm{dvol}(S^2)\wedge J_B\,. \tag{3.68}$$

Uplifting to 11d supergravity and performing a redefinition of the torus coordinates we have

$$\mathrm{d}s^2(\mathcal{M}_{11}) = \mathrm{d}s^2(\mathrm{AdS}_3) + \frac{\mathrm{d}s^2(S^2)}{4m^2} + \frac{1}{m_B^2}\left[\mathrm{d}s^2(B) + \frac{1}{\tau_2}(\mathrm{d}\tilde{x} + \tau_1\mathrm{d}\tilde{y})^2 + \tau_2\mathrm{d}\tilde{y}^2\right] \tag{3.69}$$

$$G_4 = \frac{1}{2mm_B^2}\mathrm{dvol}(S^2)\wedge(J_B + \mathrm{d}\tilde{x}\wedge\mathrm{d}\tilde{y})\,, \tag{3.70}$$

where J_B is the Kähler form on the base. We have redefined the torus coordinates to be

$$\tilde{x} = \mathrm{e}^{-2H}x\,,\quad \tilde{y} = \mathrm{e}^{-2H}y\,. \tag{3.71}$$

The periods of the two coordinates are given by

$$R_{\tilde{y}} = \frac{l_s^2}{R_{IIB}}\,,\quad R_{\tilde{x}} = \frac{l_s^2}{R_{IIB}}\,, \tag{3.72}$$

where $R_{IIB} = \frac{1}{Mm}$ is the radius of the S^1 in Type IIB which we have T-dualized along, whose coordinate has been normalised to give the canonical 2π period.

As remarked earlier, the Type IIB solution is singular over the discriminant locus where the fiber degenerates. As such the 11d supergravity metric we obtain from the explicit T-duality and uplift is only valid away from the singular loci. To make progress, we exploit the fact that the algebraic variety $\mathbb{E}_\tau \hookrightarrow Y_3 \to B$, with only I_1 singular fibers[9], is smooth and compact, and has $c_1(Y_3) = 0$, thus, by Yau's theorem, there exists a global non-singular Ricci-flat metric, of which (2.105) is an approximation valid only away from the singularities. The 11d supergravity solution

[9]As has been previously stated if Y_3 contains singularities then one can construct the smooth and compact resolution of the singularities of Y_3 and the following analysis generalises.

is therefore given by

$$\mathrm{d}s^2(\mathcal{M}_{11}) = \mathrm{d}s^2(\mathrm{AdS}_3) + \frac{1}{4m^2}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) + \frac{1}{m_B^2}\mathrm{d}s^2(Y_3) \tag{3.73}$$

$$G_4 = \frac{1}{2mm_B^2}\mathrm{dvol}(S^2) \wedge J_{Y_3}\,. \tag{3.74}$$

This solution falls within the classification of [79], specialised to elliptically fibered Calabi–Yau threefolds. Despite the fact that we do not know this metric explicitly, we will be able to compute the central charges for this solution as we discuss in section 3.4.

As commented in [79], this solution agrees locally with the geometry discussed in [80]. The M5-branes therefore wrap the 4-cycle Poincaré dual to the Kähler form J_{Y_3}, which is an ample divisor in the Calabi–Yau. Using the expansion (B.13) we see that this divisor is a linear combination of B and $\widehat{C}_\alpha$, which are divisors arising from pullbacks of curves in the base. As we only consider I_1 singularities in the fiber there are no Cartan divisors D_i. The presence of M5s wrapping the base of the Calabi–Yau is consistent with the M KK-monopoles in the Type IIB supergravity solution described in section 3.1.3. The sequence of dualities relating these two supergravity solutions is described in detail in [71]. The T-duality of M KK-monopoles in Type IIB gives rise to M NS5-branes along $\mathrm{AdS}_3 \times B$, which uplift to M M5-branes wrapped on the base. The D3-branes wrapped on the curve C in the base are uplifted to M5-branes wrapped on the elliptic surface $\widehat{C}$ as described in section 1.1.4. As noted in [71], these two stacks of M5-branes can be deformed into one stack wrapped on a linear combination of B and $\widehat{C}$ provided the curve C is sufficiently ample in the base.

3.3.2 M5-Brane Solutions

Analogous to the discussion conducted in subsection 3.1.3 we shall construct the explicit smeared brane solution which gives (3.69) in the near-horizon limit. To construct this solution one may either T-dualize the “pre near-horizon” solution obtained in (3.21) along the Hopf fiber and then uplift or use the brane smearing techniques employed previously to combine N M5-branes wrapping $\mathbb{R}^{1,1} \times \widehat{C}$ and M M5-branes wrapping $\mathbb{R}^{1,1} \times B$, in the background $\mathbb{R}^{1,1} \times \mathbb{R}^3 \times Y_3$ with $M > 0$. Both methods result in the same solution given by[10]

$$
\begin{aligned}
m_B^2 \mathrm{d}s^2(\mathcal{M}_{11}) &= \left(\frac{r+q_N}{r+q_M}\right)^{-\frac{2}{3}} \left(\frac{1}{\tau_2}(\mathrm{d}y+\tau_1\mathrm{d}\psi)^2+\tau_2\mathrm{d}\psi^2\right) && (3.75)\\
&+ \left(\frac{r+q_N}{r+q_M}\right)^{\frac{1}{3}} \left(\frac{(r+q_N)(r+q_M)}{r^2}(\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2))\right)\\
&+ \left(\frac{r+q_N}{r+q_M}\right)^{\frac{1}{3}} \left(\frac{r}{r+q_N}(-\mathrm{d}t^2+\mathrm{d}x^2)+\mathrm{d}s^2(B)\right)\\
G_4 &= \frac{1}{m_B^3}\mathrm{dvol}(S^2)\wedge(q_N J_B+q_M\mathrm{dvol}(\mathbb{E}_\tau))\,, && (3.76)
\end{aligned}
$$

where

$$
q_N = 2\pi^2 {\ell_{\mathrm{p}}}^3 m_B^3 N \quad \text{and} \quad q_M = \frac{2\pi^2 {\ell_{\mathrm{p}}}^3 m_B^3 M}{\mathrm{vol}(\mathbb{E}_\tau)}\,. \tag{3.77}
$$

Of course, as already mentioned, this solution has singularities arising from B and also τ. Notice that the Calabi–Yau metric is now warped and we are unable to resolve these singularities as in the previous subsection. However here we are interested in understanding the behaviour in the radial direction r and so we shall not discuss this issue further.

Taking the near-horizon limit one obtains the metric

$$
\begin{aligned}
m_B^2\mathrm{d}s^2(\mathcal{M}_{11})_{r\to 0} &= \left(\frac{q_N}{q_M}\right)^{\frac{1}{3}}\Bigg[\frac{r}{q_N}(-\mathrm{d}t^2+\mathrm{d}x^2)+\frac{q_N q_M}{r^2}\mathrm{d}r^2+q_N q_M(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)\\
&\quad +\mathrm{d}s^2(B)+\frac{q_M}{q_N}\left(\frac{1}{\tau_2}(\mathrm{d}y+\tau_1\mathrm{d}\psi)^2+\tau_2\mathrm{d}\psi^2\right)\Bigg]\,. && (3.78)
\end{aligned}
$$

Upon identifying the warp factor to be $\mathrm{e}^{8H}=\frac{q_N}{q_M}$, the inverse radius squared of AdS_3 to be $\frac{m_B^2}{m^2}=4q_N q_M$ and performing the change of coordinates $r=4q_N q_M\rho^2$, $y=\sqrt{\frac{q_N}{q_M}}\tilde{y}$, and $\psi=\sqrt{\frac{q_N}{q_M}}\tilde{\psi}$ one recovers (3.69) exactly and therefore an unwarped Calabi–Yau metric which may now be resolved. One also finds that the flux matches exactly with (3.70). Asymptotically, that is $r\to\infty$, the metric approaches the space $\mathbb{R}^{1,1}\times\mathbb{R}^3\times Y_3$, this is the space far away from the M5-branes. We emphasise that this geometry arises from N M5-branes wrapped on $\mathbb{R}^{1,1}\times\widehat{C}$ plus M M5-branes wrapped on $\mathbb{R}^{1,1}\times B$, with B the base of Y_3, the latter M5-branes can be seen to arise from the initial M KK-monopoles in the Type IIB solution.

One may also consider the case of N M5-branes wrapping only $\mathbb{R}^{1,1}\times\widehat{C}$ in the background $\mathbb{R}^{1,1}\times\mathbb{R}^3\times Y_3$. This is the formal definition of $M=0$. The solution of

[10]Note that q_N is the same as the constant appearing in (3.20) upon using the relation ${\ell_{\mathrm{p}}}^3=\frac{g_s l_s^4}{R_{IIB}}$ and the fact that $R_{IIB}=\frac{2}{m_B}$ for this T-duality and uplift. Recall that R_{IIB} is the radius of the S^1 in Type IIB along which we have T-dualized with the S^1 coordinate having the canonical 2π period.

this setup obtained from brane smearing is

$$
\begin{aligned}
m_B^2 \mathrm{d}s^2(\mathcal{M}_{11}) &= \left(\frac{r+q_N}{r}\right)^{-\frac{2}{3}} \left(\frac{1}{\tau_2}(\mathrm{d}y+\tau_1 \mathrm{d}\psi)^2+\tau_2 \mathrm{d}\psi^2\right) + \left(\frac{r+q_N}{r}\right)^{\frac{1}{3}} \mathrm{d}s^2(B) \qquad (3.79)\\
&+ \left(\frac{r+q_N}{r}\right)^{\frac{1}{3}} \left(\frac{r}{r+q_N}(\mathrm{d}x^2-\mathrm{d}t^2)+\frac{(r+q_N)}{r}(\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta \mathrm{d}\varphi^2))\right) ,\\
G_4 &= \mathrm{dvol}(S^2)\wedge q_N J_B \,, \qquad (3.80)
\end{aligned}
$$

with q_N as before. Notice that this agrees with taking the limit $M \to 0$ in (3.75). Recall that $\widehat{C}$ is not an ample divisor and therefore the M5-branes do *not* wrap an ample divisor as in the $M \neq 0$ case. Asymptotically the metric approaches $\mathbb{R}^{1,1}\times\mathbb{R}^3\times Y_3$ as before, however the metric is singular at $r=0$ now. To see this one computes the Ricci scalar to be[11]

$$
\mathrm{R} = \frac{q_N^2}{3r^{2/3}(r+q_N)^{10/3}} \,, \qquad (3.81)
$$

which clearly diverges at $r=0$. Upon taking the near-horizon limit one does not obtain an AdS factor, this of course matches with our previous analysis that we can only get an AdS_3 solution if the divisor wrapped by the branes is ample. Note that this does not imply that when N M5-branes wrap $\mathbb{R}^{1,1}\times\widehat{C}$, the dual 2d field theory does *not* flow to a SCFT in the IR. It just means that the IR SCFT does not have an AdS_3 gravity dual in 11d supergravity.

Recall, as discussed in section 3.1.3, that in Type IIB the $M=1$ case has two UV completions. One may consider either N D3-branes wrapping $\mathbb{R}^{1,1}\times C$ in the presence of a single KK-monopole or replacing the Taub-NUT space by flat space $\mathbb{R}^4/\mathbb{Z}_M$. Applying T-duality along the Hopf fiber of (3.25) and uplifting we obtain the 11d supergravity solution

$$
\begin{aligned}
\mathrm{d}s^2 &= (R^2+q_N)^{1/3}\Bigg(\frac{R^2}{R^2+q_N}(-\mathrm{d}t^2+\mathrm{d}x^2)+\frac{R^2+q_N}{R^2}(\mathrm{d}R^2+R^2(\mathrm{d}\theta^2+\sin^2\theta \mathrm{d}\varphi^2))\\
&\quad +\mathrm{d}s^2(B)+\frac{1}{R^2+q_N}\left(\frac{1}{\tau_2}(\mathrm{d}y+\tau_1\mathrm{d}\psi)^2+\tau_2\mathrm{d}\psi^2\right)\Bigg) \,, \qquad (3.82)\\
G_4 &= \mathrm{dvol}(S^2)\wedge\left(\frac{q_N}{4}J_B+q_M \mathrm{dvol}(\mathbb{E}_\tau)\right) \,, \qquad (3.83)
\end{aligned}
$$

with q_N and q_M as before. Of course in the near-horizon limit we obtain (3.69), however asymptotically the metric is now degenerate. This should be contrasted with the M KK-monopoles solution which has a good UV completion.[12]

[11] For ease of reading we present the result of replacing Y_3 with T^6 though the singularity persists if one reinstates the Y_3.

[12] One may, as before, consider $\mathbb{R}^4/\mathbb{Z}_M$ in place of $\mathbb{R}^4$, however similarly one obtains a degenerate UV completion.

To summarise, in this section we have found the "pre near-horizon" solution to the 11d supergravity AdS_3 solution (3.69). One may obtain such a near-horizon solution from two 11d supergravity solutions, both can be seen as the solution arising from a T-duality along a Hopf fiber and uplift of a Type IIB solution, (3.21) and (3.25) respectively. The solution arising from M KK-monopoles has a good UV completion whilst the solution arising from no KK-monopoles has a degenerate UV completion.

3.3.3 Flux Quantisation

For an 11d supergravity solution to be well-defined one must quantize the fluxes through all integral cycles in the geometry. Following [81], the correct quantization condition to impose is that for all $\Sigma_4 \in H_4(\mathcal{M}_{11}, \mathbb{Z})$,

$$n(\Sigma_4) = \int_{\Sigma_4} \left[\frac{1}{(2\pi \ell_\mathrm{p})^3} G_4 - \frac{p_1}{4} \right] \in \mathbb{Z} \,, \tag{3.84}$$

where ℓ_p is the eleven-dimensional Planck length and p_1 is the first Pontryagin class of $\mathcal{M}_{11}$ defined as

$$p_1 = -\frac{1}{8\pi^2} \mathrm{Tr}[\mathcal{R}^2] \,. \tag{3.85}$$

There are two types of integral four-cycles in $\mathcal{M}_{11}$ to consider: the divisors D in the Calabi–Yau threefold Y_3 as summarised in section B.2.1, and the four-cycles $S^2 \times \mathbb{E}_\tau$ and $S^2 \times C_\alpha$ with C_α, as before, forming a basis of $H_2(B, \mathbb{Z})$.

We shall first consider the contributions from the $p_1/4$ term and show that they are all integral. As the metric is a product space we have

$$\mathcal{R}_{\mathcal{M}_{11}} = \mathcal{R}_{\mathrm{AdS}_3} + \mathcal{R}_{S^2} + \mathcal{R}_{Y_3} \,. \tag{3.86}$$

where $\mathcal{R}^a{}_b = \frac{1}{2} R^a{}_{b\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. In particular, p_1 is non-trivial only on the Calabi–Yau, thus $p_1(\mathcal{M}_{11}) = -2c_2(Y_3)$, which is given in (B.19). This implies that the $p_1/4$ term integrated over the four-cycles $S^2 \times \mathbb{E}_\tau$ and $S^2 \times C_\alpha$ vanishes. On the other hand, the integral of $c_2(Y_3)$ over every divisor D is always divisible by two, as shown in (B.27),

$$\int_D c_2(Y_3) = 2(h^{1,1}(D) - 4h^{0,2}(D) + 2h^{0,1}(D) - 4) \,, \tag{3.87}$$

therefore the flux quantization condition reduces simply to

$$n(\Sigma_4) = \frac{1}{(2\pi \ell_\mathrm{p})^3} \int_{\Sigma_4} G_4 \in \mathbb{Z} \,. \tag{3.88}$$

The form of the G_4-flux implies that the quantization over the divisors of Y_3 is trivial, $n(D) = 0$, and therefore the relevant four-cycles to perform the quantization over are $S^2 \times \mathbb{E}_\tau$ and $S^2 \times C_\alpha$. Then we have

$$n(S^2 \times C_\alpha) \;=\; \frac{2\pi \mathrm{e}^{4H}}{m m_B^2 (2\pi \ell_{\mathrm{p}})^3} \int_{C_\alpha} J_{Y_3} \,, \tag{3.89}$$

where J_{Y_3} is given in (B.13). Recalling that $\int_{C_\alpha} J_{Y_3} = k_\alpha \in \mathbb{Z}^+$, we see that imposing the condition

$$\mathbb{Z}^+ \ni \widetilde{N} = \frac{2\pi \mathrm{e}^{4H}}{m m_B^2 (2\pi \ell_{\mathrm{p}})^3} \,, \tag{3.90}$$

guarantees that $n(S^2 \times C_\alpha)$ is correctly quantized. For later, we shall also need the volume of the elliptic fibration. This is constant over the base. We define the integer $\widetilde{M}$ as

$$\widetilde{M} = \widetilde{N} \int_{\mathbb{E}_\tau} J_{Y_3} = \widetilde{N} k^0 \,, \tag{3.91}$$

that is $\mathrm{vol}(\mathbb{E}_\tau) = \frac{\widetilde{M}}{\widetilde{N}}$. We shall show that $\widetilde{M} = M$ where the latter M is that arising in Type IIB from the Lens space quotient. To see this we must use the periods of the elliptic fiber coordinates arising from the Type IIB solution, (3.72). As the volume is constant over the base we may compute it away from any singularity. We find

$$\mathrm{vol}(\mathbb{E}_\tau) = (2\pi)^2 R_{\tilde{x}} R_{\tilde{y}} = (2\pi)^2 m m_B^2 M \mathrm{e}^{-4H} {\ell_{\mathrm{p}}}^3 = \frac{M}{\widetilde{N}} \,, \tag{3.92}$$

where we have used the relation

$${\ell_{\mathrm{p}}}^3 = \frac{g_{\mathrm{s}} l_s^4}{R_{IIB}} \,, \tag{3.93}$$

which follows from the T-duality and uplift. Using this relation we may also show that the N in Type IIB is the same as the $\widetilde{N}$ in 11d supergravity. Observe that

$$N = \frac{16\pi^2 \mathrm{e}^{4H}}{4 m^2 m_B^2 M (2\pi \ell_{\mathrm{s}})^4 g_{\mathrm{s}}} = \frac{2\pi \mathrm{e}^{4H}}{m m_B^2 (2\pi \ell_{\mathrm{p}})^3} = \widetilde{N} \,. \tag{3.94}$$

We conclude that the two integers appearing in Type IIB and 11d supergravity solutions can be identified, namely $N = \widetilde{N}$ and $M = \widetilde{M}$. For notational clarity we shall drop the tildes from now on as there is no confusion. We remark that in Type IIB M corresponds to the number of KK-monopoles in the geometry whilst in 11d supergravity it is proportional to the volume of the elliptic fibration.

3.4 Holographic Central Charges from M-theory

3.4.1 Leading Order Central Charges

The gravitational central charge for the 11d supergravity solution $\mathrm{AdS}_3 \times S^2 \times Y_3$ was computed in [82]. We reproduce it here for completeness using (B.6)

$$
\begin{aligned}
(c_L^{11})^{(3)} = (c_R^{11})^{(3)} = c^{11}_{\text{sugra}} &= \frac{3}{2mm_B^2}\frac{2^5\pi^2}{(2\pi\ell_{\mathrm{p}})^9}\int_{M_8}\frac{1}{4m^2m_B^4}\mathrm{dvol}(S^2)\wedge\mathrm{dvol}(Y_3)\\
&= \frac{3\pi^3 2^4}{((2\pi\ell_{\mathrm{p}})^3 mm_B^2)^3}\int_{Y_3}\frac{1}{6}J_{Y_3}\wedge J_{Y_3}\wedge J_{Y_3}\\
&= N^3\, C_{IJK}k^I k^J k^K\,,
\end{aligned}
\tag{3.95}
$$

where we have expanded the Kähler form in a basis of $(1,1)$-forms on the Calabi–Yau threefold as in (B.13) and C_{IJK} are the triple intersection numbers as given in section B.2.1, with $I=0$ included in this expansion. This result, as noted in [82], matches the original field theory computation in [80] and [83].

The Kähler form is expanded as in (B.13), where the coefficient in front of the zero-section ω_0 is

$$
k_0 = \mathrm{vol}(\mathbb{E}_\tau) = \frac{M}{N}\,,
\tag{3.96}
$$

the volume of the elliptic fiber. The central charge (3.95) can then be expanded into three terms

$$
\begin{aligned}
(c_L^{11})^{(3)} = (c_R^{11})^{(3)} &= N^3\left(3k_0k_\alpha k_\beta\int_{Y_3}\omega_0\wedge\omega_\alpha\wedge\omega_\beta + 3k_0^2k_\alpha\int_{Y_3}\omega_0\wedge\omega_0\wedge\omega_\alpha + k_0^3\int_{Y_3}\omega_0^3\right)\\
&= N^3\left(3k_0k_\alpha k_\beta\int_B\omega_\alpha\wedge\omega_\beta - 3k_0^2k_\alpha\int_B c_1(B)\wedge\omega_\alpha + k_0^3\int_B c_1(B)^2\right)\\
&= 3N^2MC\cdot C - 3NM^2c_1(B)\cdot C + M^3(10-h^{1,1}(B))\,,
\end{aligned}
\tag{3.97}
$$

where we have made use of (B.17).

3.4.2 Chern-Simons Terms and $c_L^{11} - c_R^{11}$

We now calculate $c_L^{11} - c_R^{11}$ by using the eight derivative corrections as presented in [84]. The term that will be relevant for us is the Chern-Simons term [85]

$$
S_{CS} = -\frac{(4\pi\kappa_{11})^{2/3}}{2\kappa_{11}^2}\int_{M_{11}} C_3\wedge X_8\,,
\tag{3.98}
$$

where

$$
X_8 = \frac{1}{(2\pi)^4 2^6\cdot 3}\left(\mathrm{Tr}[\mathcal{R}^4] - \frac{1}{4}(\mathrm{Tr}[\mathcal{R}^2])^2\right)\,.
\tag{3.99}
$$

We wish to dimensionally reduce this to obtain Chern-Simons terms in the 3d action. From the coefficient in front of the 3d Chern-Simons term one can extract $c_L^{11} - c_R^{11}$ by using (B.7). Using (3.86) one can see that $\mathrm{Tr}[\mathcal{R}^4] = 0$. We wish to find the term proportional to (B.7) and so we shall drop terms that do not contribute to this if necessary

$$\begin{aligned} S_{CS} &= \frac{(4\pi\kappa_{11})^{2/3}}{2^3\kappa_{11}^2}\frac{1}{(2\pi)^4 2^6\cdot 3}\int_{M_{11}} C_3\wedge \mathrm{Tr}[\mathcal{R}^2]\wedge \mathrm{Tr}[\mathcal{R}^2] \\ &= \frac{N\pi}{2^4\cdot 3(2\pi)^2}\int_{Y_3} J_{Y_3}\wedge c_2(Y_3)\int_{\mathrm{AdS}_3}\omega_{CS}(\Gamma_{\mathrm{AdS}_3})\,, \end{aligned} \tag{3.100}$$

from which we obtain

$$c_L^{11} - c_R^{11} = \frac{N}{2}\int_{Y_3} J_{Y_3}\wedge c_2(Y_3)\,, \tag{3.101}$$

which is in agreement with [82].[13]

To evaluate (3.101) we use the expansion of the Kähler form in (B.13) and the form of $c_2(Y_3)$ as in (B.19). With this information we reduce the integrals in (3.101) to integrals over the base of the fibration, namely[14]

$$\int_{Y_3} c_2(Y_3)\wedge\omega_0 = \int_B c_2(B) - c_1(B)^2 = 2h^{1,1}(B) - 8\,. \tag{3.102}$$

For the remaining term, the Poincaré dual to ω_α are divisors $D_\alpha = \widehat{C}_\alpha$ which are pull-backs of curves in the base. Thus the integral over Y_3 is only non-vanishing for those terms in $c_2(Y_3)$, which have fiber components, *i.e.* the $12\omega_0\wedge c_1(B)$ term, which leads to

$$\int_{Y_3} c_2(Y_3)\wedge\omega_\alpha = 12c_1(B)\cdot C_\alpha\,. \tag{3.103}$$

Combining these terms we find

$$c_L^{11} - c_R^{11} = 6Nc_1(B)\cdot C + M(h^{1,1}(B) - 4)\,. \tag{3.104}$$

3.4.3 Chern-Simons Couplings from 11d Supergravity

The 11d supergravity solution $\mathrm{AdS}_3\times S^2\times Y_3$ has dual SCFTs with small $\mathcal{N} = (0,4)$ superconformal symmetry. In order to determine the left and right central charges one must also calculate the level k_r of the superconformal $SU(2)_r$ R-symmetry at sub-leading order. The leading and sub-leading corrections to the level k_r were

[13] Note that $\mathrm{Tr}[\mathcal{R}^2] = 16\pi^2 c_2(Y_3)$ which is valid for a Calabi–Yau threefold whilst working in real coordinates with the normalisation as in [86].

[14] Note the integral $\int_{Y_3}$ can be translated into one over B by using the intersection ring relations, and extracting the coefficient of σ^2.

computed in [76, 82] to be

$$k_r = \frac{N^3}{6} C_{IJK} k^I k^J k^K + \frac{N}{12} \int_{Y_3} J_{Y_3} \wedge c_2(Y_3) \,. \tag{3.105}$$

These terms are computed by deforming the metric on the two-sphere to contain connections which depend on AdS_3 only

$$\mathrm{d}s^2(S^2) \to (\mathrm{d}x^a - A^{ab} x^b)(\mathrm{d}x^a - A^{ac} x^c) \,, \tag{3.106}$$

where $\sum_{a=1}^{3} (x^a)^2 = 1$. These connections are identified with the $SO(3)$ gauge fields for the R-symmetry.

The leading order term is computed from the 11d term

$$S_{AFF} = -\frac{1}{12\kappa_{11}^2} \int A_3' \wedge G_4' \wedge G_4' \,, \tag{3.107}$$

where we have used the conventions of [84]. For the deformed metric the fluxes are corrected by terms involving the R-symmetry gauge fields and are given by

$$A_3' = \frac{2\pi \mathrm{e}^{4H}}{m m_B^2} e_1^{(0)} \wedge J_{Y_3} \tag{3.108}$$

$$G_4' = \frac{2\pi \mathrm{e}^{4H}}{m m_B^2} e_2 \wedge J_{Y_3} \,, \tag{3.109}$$

where e_2 is the unique two-form for the S^2 bundle satisfying $\int_{S^2} e_2 = 1$ and $\mathrm{d}e_2 = 0$. The one-form $e_1^{(0)}$ is defined by $\mathrm{d}e_1^{(0)} = e_2$. The overall factors in G_4' have been fixed by requiring that the quantization pre-deformation is the same as that post-deformation. Inserting these expressions into (3.107) we obtain

$$S_{AFF} = -\frac{(2\pi)^3 \mathrm{e}^{12H}}{12\kappa_{11}^2 m^3 m_B^6} \int_{Y_3} J_{Y_3} \wedge J_{Y_3} \wedge J_{Y_3} \int_{\mathrm{AdS}_3 \times S^2} e_1^{(0)} \wedge e_2 \wedge e_2 \,. \tag{3.110}$$

To simplify this expression we make use of the formula derived in [76]

$$\int_{\mathrm{AdS}_3 \times S^2} e_1^{(0)} \wedge e_2 \wedge e_2 = -\frac{1}{2(2\pi)^2} \int_{\mathrm{AdS}_3} \omega_{CS}(A) \,, \tag{3.111}$$

which originates from [87]. Recalling the expression $N = \frac{2\pi}{(2\pi l_p)^3 m m_B^2}$ we obtain

$$S_{AFF} = \frac{\pi \mathrm{e}^{12H}}{12\kappa_{11}^2 m^3 m_B^6} \int_{Y_3} J_{Y_3} \wedge J_{Y_3} \wedge J_{Y_3} \int_{\mathrm{AdS}_3} \omega_{CS}(A) \tag{3.112}$$

$$= \frac{N^3}{24\pi} C_{IJK} k^I k^J k^K \int_{\mathrm{AdS}_3} \omega_{CS}(A) \,, \tag{3.113}$$

The level k_r is extracted from the coefficient of the Chern-Simons term from the definition in (3.53). From this we obtain the leading order term in (3.105).

The sub-leading order term is found by computing S_{CS} for the deformed metric, which now contains a contribution from the R-symmetry gauge fields

$$S_{CS} = \frac{N}{192\pi} \int_{CY} J_{Y_3} \wedge c_2(Y_3) \left(\int_{\mathrm{AdS}_3} \omega_{CS}(\Gamma_{\mathrm{AdS}_3}) + 4 \int_{\mathrm{AdS}_3} \omega_{CS}(A) \right) , \tag{3.114}$$

where the trace in $\omega_{CS}(A)$ is taken over the fundamental representation of $SU(2)$. The factor of 4 appearing in the gauge Chern-Simons term arises from changing the trace from over the vector representation of $SO(3)$ to $SU(2)$ fundamental. Comparing (3.114) to (3.53) the sub-leading term indeed matches that in (3.105).

Using the results from section 3.4.2 the level can be expressed as

$$\begin{aligned} k_r &= \frac{N^3}{6} C_{IJK} k^I k^J k^K + \frac{N}{12} \int_{Y_3} c_2(Y_3) \wedge J_{Y_3} && (3.115) \\ &= \frac{1}{2} N^2 M C \cdot C + \frac{N}{2}(2 - M^2) c_1(B) \cdot C + \frac{M^3}{6}(10 - h^{1,1}(B)) + \frac{M}{6}(h^{1,1}(B) - 4) \, . && (3.116) \end{aligned}$$

The left and right central charges can now be deduced by using the relation $c_R^{11} = 6k_r$ [88]. We obtain the central charges

$$\begin{aligned} c_R^{11} &= 3N^2 M C \cdot C + 3N(2 - M^2) c_1(B) \cdot C + M^3(10 - h^{1,1}(B)) + M(h^{1,1}(B) - 4) \, , && (3.117) \\ c_L^{11} &= 3N^2 M C \cdot C + 3N(4 - M^2) c_1(B) \cdot C + M^3(10 - h^{1,1}(B)) + 2M(h^{1,1}(B) - 4) \, . && (3.118) \end{aligned}$$

Interestingly, we note that the right-moving central charge c_R^{11} can be shown to be an integer multiple of 6 as expected [20]. To see this we rewrite c_R^{11} as

$$\begin{aligned} c_R^{11} &= 6N^2 M(g-1) + (6N + 3NM(N-M))\, c_1(B) \cdot C && (3.119) \\ &\quad + 6M^3 + (M-1)M(M+1)(4 - h^{1,1}(B)) \, . && (3.120) \end{aligned}$$

It is an elementary exercise to show that each term in the expression above is indeed a multiple of 6, for arbitrary values of $N, M \in \mathbb{Z}$. We regard this as a non-trivial check on the interpretation of c_R^{11} as the right-moving central charge of a $(0,4)$ SCFT with small superconformal algebra.

In section 3.2 for $M > 1$ we were only able to determine the leading order central charge. To the contrary here, we have the all order expression. It would be very interesting to extend the analysis in section 3.2.3 to include $M > 1$ from the Type IIB perspective and to compare with the above expression. The recent work of [68] obtained our M-theory central charges from 5d gauged supergravity and one should be able to use their work to trace the complementary computation in Type IIB.

3.5 Field Theory: Central Charges from Anomalies and Comparisons

In this section we shall determine the central charges of the 2d SCFTs microscopically, using a UV description in terms of world-volume theories on wrapped branes. To determine these we will essentially need to compute only the anomaly polynomials of the corresponding branes, although we will discuss some subtleties involved in these computations. This complements and extends the central charge computation in section 1.1.4 from the dimensional reduction of the abelian $\mathcal{N} = 4$ SYM theory. Below we will invert the order of presentation with respect to the previous sections as we find it more convenient to begin with the M5-branes in the M-theory picture and address the D3-branes in the F-theory picture after. We also include a section summarising the results of the computations in the different setups and their comparison.

3.5.1 Anomalies from M5-branes

In this section we wish to determine the anomaly polynomial associated to the $(0, 4)$ theory on the worldvolume of the string in 5d arising from a stack of M5-branes wrapping a compact 4-cycle in a Calabi–Yau threefold.

A single M5-brane has an anomaly [89] from the chiral modes living on the 6d worldvolume of the brane; this anomaly must be cancelled by anomaly inflow from the M-theory bulk. In [90] a certain deformation of the cubic Chern-Simons term in M-theory was found to cancel the anomaly from a single M5-brane, and this was generalised in [83] to compute the total anomaly polynomial of the 6d worldvolume theory on a stack of N M5-branes. The anomaly polynomial is

$$I_8[N] = N I_8[1] + \frac{1}{24}(N^3 - N) p_2(\mathcal{N}) \,, \tag{3.121}$$

where

$$I_8[1] = \frac{1}{48}\left[p_2(\mathcal{N}) - p_2(W) + \frac{1}{4}(p_1(W) - p_1(\mathcal{N}))^2\right] , \tag{3.122}$$

is the anomaly polynomial for the free abelian tensor multiplet that lives on the worldvolume of a single M5-brane and W, $\mathcal{N}$ are respectively the 6d submanifold the M5-brane wraps, and the normal, or $SO(5)$ R-symmetry, bundle associated to the transverse directions of the M5-brane worldvolume in the 11d spacetime.

The theory living on the worldvolume of N M5-branes in flat space is the interacting $(2, 0)$ superconformal field theory of type A_{N-1} coupled to the free abelian tensor multiplet. We can determine the anomaly polynomial of the A_{N-1} theory by subtracting off the contribution from the latter,

$$I_8^{\text{int}}[N] = (N-1)I_8[1] + \frac{1}{24}(N^3 - N)p_2(\mathcal{N})\,. \tag{3.123}$$

This agrees with [91] where the anomaly polynomial of the 6d $(2,0)$ theories associated to ADE Lie algebras was conjectured to be

$$I_8(G) = r(G)I_8[1] + \frac{1}{24}d(G)h^\vee(G)p_2(\mathcal{N})\,, \tag{3.124}$$

where r, d, and $h^\vee$ are the rank, dimension, and the dual Coxeter number of the ADE group G, respectively.

Following [83] the anomaly polynomial I_4 for the string arising from the M5-brane wrapping a compact surface P inside a Calabi–Yau threefold[15], Y_3, can be determined by integrating the 6d anomaly polynomial over P. For such an M-theory setup the tangent and normal bundles decompose as

$$\begin{aligned} TW &= TP \oplus TW_2\,, \\ \mathcal{N} &= \mathcal{N}_{P/Y_3} \oplus \mathcal{N}_3\,, \end{aligned} \tag{3.125}$$

where W_2 is the worldvolume of the string, $\mathcal{N}_{P/Y_3}$ is the normal bundle of P inside of the Calabi–Yau, and $\mathcal{N}_3$ is the bundle associated to the $SO(3)_T$ global symmetry from the rotations of the 3 transverse directions to the string in 5d. Under these bundle decompositions the Pontryagin classes decompose, via the splitting principle, to

$$\begin{aligned} p_1(\mathcal{N}_{P/Y_3} \oplus \mathcal{N}_3) &= p_1(\mathcal{N}_{P/Y_3}) + p_1(\mathcal{N}_3) \\ p_2(\mathcal{N}_{P/Y_3} \oplus \mathcal{N}_3) &= p_2(\mathcal{N}_{P/Y_3}) + p_2(\mathcal{N}_3) + p_1(\mathcal{N}_{P/Y_3})p_1(\mathcal{N}_3)\,, \end{aligned} \tag{3.126}$$

and similarly for $p_i(W)$.

First, let us consider the integration of the anomaly polynomial of a single M5-brane:

$$48\int_P I_8[1] = 2p_1(\mathcal{N}_3)\int_P p_1(\mathcal{N}_{P/Y_3}) - \frac{1}{2}(p_1(W_2) + p_1(\mathcal{N}_3))\int_P (p_1(P) + p_1(\mathcal{N}_{P/Y_3}))\,. \tag{3.127}$$

We can use the adjunction formula

$$TY_3 = TP \oplus \mathcal{N}_{P/Y_3}\,, \tag{3.128}$$

to rewrite the last integrand as $p_1(Y_3)$. Finally we can use the representation of the Pontryagin classes in terms of the Chern classes,

$$p_1(Y_3) = -2c_2(Y_3) + c_1(Y_3)^2\,, \tag{3.129}$$

[15]Note that the Calabi–Yau threefold does not, at this point, need to be elliptically fibered. Moreover, P does not have to be a (very) ample divisor.

and the Calabi–Yau property of Y_3, $c_1(Y_3) = 0$, to rewrite the two integrands in terms of the Chern classes of P and Y_3. In conclusion, the integral over the total anomaly polynomial I_8, combining both the free and interacting theories living on the M5-brane, is [83]

$$I_4[N] = \int_P I_8[N] = N I_4[1] + \frac{1}{24}(N^3 - N)P^3 p_1(\mathcal{N}_3)\,, \tag{3.130}$$

where we have rewritten the integrals over P as integrals over Y_3 using intersection notation, and

$$I_4[1] = \int_P I_8[1] = \frac{1}{48}\left[2P^3 p_1(\mathcal{N}_3) + c_2(Y_3) \cdot_{Y_3} P(p_1(W_2) + p_1(\mathcal{N}_3))\right] . \tag{3.131}$$

The gravitational anomaly determines the difference between the left- and right-moving central charges of the $(0,4)$ SCFT on the string, and can be read off from the anomaly

$$I_4 \supset \frac{c_L - c_R}{24} p_1(W_2)\,. \tag{3.132}$$

Thus we immediately determine that

$$c_L - c_R = \frac{1}{2} N c_2(Y_3) \cdot_{Y_3} P\,. \tag{3.133}$$

From the anomaly polynomial it is also possible to read off the level associated to the $SO(3)_T$ global symmetry by studying the $k_3 p_1(\mathcal{N}_3)/4$ term. We find

$$k_3 = \frac{1}{6} N^3 P^3 + \frac{1}{12} N c_2(Y_3) \cdot_{Y_3} P\,. \tag{3.134}$$

For future reference we also note that k_3 can be expressed directly in terms of the Hodge numbers of P. Using the expansion of the Chern numbers in terms of the Hodge numbers we have

$$\begin{aligned} P^3 &= 10h^{0,2}(P) - 8h^{0,1}(P) - h^{1,1}(P) + 10 \\ c_2(Y_3) \cdot_{Y_3} P &= 2h^{1,1}(P) - 8h^{0,2}(P) + 4h^{0,1}(P) - 8\,. \end{aligned} \tag{3.135}$$

At this point we shall specialise to considering that Y_3 is an elliptic fibration. From the Shioda–Tate–Wazir theorem as described in section B.2.1 we know the divisors in Y_3 that generate the Neron–Severi lattice, and we would like to compute these quantities, $c_L - c_R$ and k_3, for representatives of certain linear systems of these divisors on Y_3. Recall that we are interested in elliptically fibered Calabi–Yau threefolds $\pi : Y_3 \to B$, with section, and that the two types of basis divisors of principle interest are the base, B, and the pullbacks of curves in the base, $\widehat{C}_\alpha = \pi^* C_\alpha$, such that the curve is not contained inside the discriminant locus of the elliptic fibration.

Let us consider an M5-brane wrapping a smooth irreducible divisor in the linear system

$$D \in |MB + N\widehat{C}|\,, \tag{3.136}$$

where $\widehat{C}$ is a linear combination of the $\widehat{C}_\alpha$, and compute the above quantities for $P = D$. The cohomology class of D can be written as

$$[D] = M[B] + N[\widehat{C}]\,, \tag{3.137}$$

and thus the first intersection number that must be computed is

$$\begin{aligned}[D]^3 &= M^3[B]^3 + N^2[\widehat{C}]^3 + 3M^2N[B] \cdot_{Y_3} [B] \cdot_{Y_3} [\widehat{C}] + 3MN^2[B] \cdot_{Y_3} [\widehat{C}] \cdot_{Y_3} [\widehat{C}] \\ &= M^3(10 - h^{1,1}(B)) + 3M^2N(-c_1(B) \cdot_B C) + 3MN^2 C \cdot_B C\,,\end{aligned} \tag{3.138}$$

where for the final two intersections we have used the triple intersection numbers for elliptic Calabi–Yau varieties of section B.2.1. Furthermore

$$\begin{aligned}\frac{1}{2}c_2(Y_3) \cdot_{Y_3} [D] &= \frac{1}{2}Mc_2(Y_3) \cdot_{Y_3} [B] + \frac{1}{2}Nc_2(Y_3) \cdot_{Y_3} [\widehat{C}] \\ &= M(h^{1,1}(B) - 4) + 6Nc_1(B) \cdot_B C\,.\end{aligned} \tag{3.139}$$

Therefore we have determined that for an M5-brane wrapping an arbitrary divisor D belonging to such a linear system

$$c_L - c_R = 6Nc_1(B) \cdot C + M(h^{1,1}(B) - 4)\,, \tag{3.140}$$

and

$$\begin{aligned}k_3 = &\frac{1}{2}MN^2 C \cdot C + \frac{1}{2}N(2 - M^2)c_1(B) \cdot C \\ &+ \frac{1}{6}\left(M^3(10 - h^{1,1}(B)) + M(h^{1,1}(B) - 4)\right)\,.\end{aligned} \tag{3.141}$$

Note that to compute these coefficients we had to use the anomaly polynomial for a single M5-brane, $I_4[1]$, as M and N may be coprime, however when either M or N vanishes we see the correct result for multiple M5-branes wrapping a single divisor as in (3.130)[16].

At this point we have determined the difference in left- and right-moving central charges and the anomaly coefficient for the $SO(3)_T$ normal bundle anomaly for an the 2d $(0, 4)$ theory on the worldvolume of the string from an M5-brane wrapping an arbitrary divisor D in Y_3. From [80] it is known that if D is a very ample divisor in Y_3 then the computation of k_3 is a suitable substitute for the computation of k_r, the level of the superconformal $SU(2)_r$ R-symmetry in the IR, and thus one

[16] For arbitrary values of M and N one can consider the anomaly of a single M5-brane wrapping either the divisor D as in (3.131), or one can factor D as $D = \gcd(M, N)D'$, and consider $\gcd(M, N)$ M5-branes wrapping the divisor D' as in (3.130), by computing $I_4[\gcd(M, N)]$ for the divisor D'. It is straightforward to verify that both approaches produce the same result.

can compute the right-moving central charge through the superconformal algebra relation

$$c_R = 6k_r\,. \tag{3.142}$$

In fact, when D is ample the existence of an 11d supergravity dual of the type $\mathrm{AdS}_3 \times S^2 \times Y_3$ guarantees that $SO(3)_T$ can be identified exactly[17] with the $SU(2)_r$ R-symmetry rotating the S^2. Thus $c_R = 6k_r = 6k_3$ is valid more generally for an ample divisor D.

From the information just described it is possible to compute the left- and right-moving central charges for the $(0,4)$ SCFT living on the string from a stack of M5-branes wrapping a compact complex surface inside a Calabi–Yau threefold, assuming that the surfaces satisfy sufficient topological properties that the level associated to the superconformal R-symmetry, k_r, is the same as k_3. For a divisor D inside the linear system that we are interested in, $|MB+N\widehat{C}|$, a discussion of exactly when this divisor may be ample in Y_3 is contained in appendix B.2.3. A necessary condition for D to be an ample divisor is that

$$D \cdot C = (N-M)C\cdot C + M(2g-2) > 0\,, \tag{3.143}$$

as pointed out in (B.35). It is clear that such an inequality cannot be satisfied for arbitrary values of M, N, and g, however in the large N limit, where $N \gg M$, and when C is ample in the base, this is always satisfied. For any ample D, which then satisfies this inequality, we can use (3.140) and (3.141), to compute the right- and left-moving central charges on the M5-brane wrapping D and we find

$$\begin{aligned} c_R &= 3N^2MC\cdot C + 3N(2-M^2)c_1(B)\cdot C + M^3(10-h^{1,1}(B)) + M(h^{1,1}(B)-4)\,,\\ c_L &= 3N^2MC\cdot C + 3N(4-M^2)c_1(B)\cdot C + M^3(10-h^{1,1}(B)) + 2M(h^{1,1}(B)-4)\,. \end{aligned} \tag{3.144}$$

To determine these central charges we have used that the level k_3 of the $SO(3)_T$ normal bundle anomaly is the same as the level of the superconformal R-symmetry anomaly, however this only holds if D is ample in Y_3, which is exactly the requirement for when a supergravity dual of this 2d theory exists. From the field theory side we are justified in considering a setup where $M=0$ and we just have a stack of N M5-branes wrapping the elliptic surface $\widehat{C}$. In appendix B.2.3 we show that $\widehat{C}$ is never itself an ample divisor, but in such a situation we would like to be able to determine a prescription for computing the central charge of the $(0,4)$ theory for such a stack of M5-branes, applicable even when the divisor wrapped by the M5-branes is not ample. This will correspond to the Type IIB/D3-brane setup where there are no

[17]More specifically the $SO(3)_T$ acting on the fields of the interacting SCFT, is then exactly the $SU(2)_r$ superconformal R-symmetry of that interacting SCFT. One can see directly from the spectrum that only after the universal centre-of-mass hypermultiplet is separated out is the $SO(3)_T$ consistent with the superconformal algebra.

KK-monopoles. We postpone this discussion for M5-branes until section 3.5.3, while we now turn to the F-theory picture for this setup.

3.5.2 Anomalies of 6d Self-dual Strings

A stack of N M5-branes wrapping an elliptic surface $\widehat{C}$ inside an elliptic Calabi–Yau threefold is T-dual to a stack of N D3-branes wrapping a curve in the base of the elliptic Calabi–Yau. Such D3-brane stacks give rise to self-dual strings in 6d, and the anomaly polynomial for such strings was determined via inflow from the 6d theory in [92, 93] and extended to include arbitrary genus curves in [18]. We will assume that the curve, C, on which the D3-branes wrap has only transversal intersections with the discriminant locus of the elliptic fibration. The $(0, 4)$ worldvolume theory on the string has the global symmetry group

$$SU(2)_R \times SU(2)_L \times SU(2)_I \,, \tag{3.145}$$

where $SO(4)_T \cong SU(2)_R \times SU(2)_L$ is the rotation group to the non-compact directions transverse to the string and $SU(2)_I$ is the R-symmetry group of the 6d theory. The $SO(4)_R$ UV R-symmetry group for the $(0, 4)$ theory on the worldvolume of the string is $SU(2)_R \times SU(2)_I$.

In [92,93] the anomaly polynomial for a self-dual string, of charges Q_i with respect to the two-form potentials B_i, with dB_i self-dual, in a 6d $\mathcal{N} = (1, 0)$ theory was determined by applying a similar analysis as that was introduced in [83], and which was used in section 3.5.3 for the anomaly polynomial on a stack of M5-branes. The translation of the charges Q_i of the strings into the curve classes from the interpretation of the strings as coming from D3-branes wrapping the curve C was included in [18]. The final result for the anomaly polynomial, I_4, of the string in terms of the characteristic classes of the bundles associated to the symmetry groups (3.145) is

$$\begin{aligned} I_4 = c_2(R)\left[\frac{1}{2}N^2 C\cdot C + \frac{1}{2}Nc_1(B)\cdot C\right] + c_2(L)\left[-\frac{1}{2}N^2 C\cdot C + \frac{1}{2}Nc_1(B)\cdot C\right] \\ + c_2(I)\,[N] - \frac{1}{24}p_1(T)\,[6Nc_1(B)\cdot C] \,, \end{aligned} \tag{3.146}$$

where we have ignored contributions from any additional global (flavour) symmetries other than those discussed above, and where we recall that the genus of the curve is contained inside the above expressions implicitly via adjunction (3.37). First we can determine the difference between the left- and right-moving central charges from the gravitational anomaly term

$$c_L - c_R = 6Nc_1(B)\cdot C \,. \tag{3.147}$$

One can also read off from the anomaly polynomial the levels of the $SU(2)_{R,L,I}$ global symmetries

$$\begin{aligned} k_R &= \frac{1}{2}N^2 C\cdot C + \frac{1}{2}Nc_1(B)\cdot C \\ k_L &= -\frac{1}{2}N^2 C\cdot C + \frac{1}{2}Nc_1(B)\cdot C \\ k_I &= N\,. \end{aligned} \tag{3.148}$$

Note that the $SU(2)_r$ superconformal R-symmetry can in principle be a mix [94] of the two $SU(2)$ factors in the $SO(4)_R$ UV R-symmetry. We observe from the spectrum for $N=1$ that the IR R-symmetry for the SCFT must be $SU(2)_R$ as this is the only factor under which the bosons of all the hypermultiplets constituting the theory are uncharged. Moreover, in the next subsection we will argue (using only the information on k_L from this section) that the correct R-symmetry in the IR should be simply $SU(2)_R$ for any N. Thus there is no mixing with $SU(2)_I$ and we conclude that

$$c_R \;=\; 6k_R = 3N^2 C\cdot C + 3Nc_1(B)\cdot C\,, \tag{3.149}$$

and from (3.147) we also obtain

$$c_L = 3N^2 C\cdot C + 9Nc_1(B)\cdot C\,. \tag{3.150}$$

3.5.3 Anomaly from M5-branes on $\widehat{C}$

Let us now return to the M5-brane anomaly inflow, in the case that the branes wrap the elliptic surface $\widehat{C}$ in Y_3, which is not ample. We recall that in this instance there does not exist an AdS_3 dual, without three-form flux, because of the lack of ampleness of the divisor. However, in this section, one shall see that it is still possible to determine the central charges of the SCFT. We can immediately see from a study of the spectrum of a single M5-brane [18] that k_3 is not a suitable substitute computation for k_r when the wrapped divisor is not ample[18]. Let us first consider an arbitrary divisor P inside an arbitrary Calabi–Yau threefold. We can read off from the expressions in terms of Hodge numbers in (3.135) that

$$k_3 = h^{0,2}(P) - h^{0,1}(P) + 1\,, \tag{3.151}$$

but a direct computation of the right-moving central charge from the spectrum reveals that

$$k_r = h^{0,2}(P) + 1\,. \tag{3.152}$$

This is consistent, as for P an ample divisor inside a Calabi–Yau threefold then $h^{0,1}(P) = 0$ by the Lefschetz hyperplane theorem.

[18]This puzzle was raised in [95, 96].

Now, let us consider multiple (N) M5-branes wrapping the divisor $P = \widehat{C}$; hence $M = 0$ in the notation of section 3.5.1. Using standard mathematical results for the cohomologies of elliptic surfaces

$$\begin{aligned} h^{0,2}(\widehat{C}) &= \frac{1}{2}\left(C \cdot C + c_1(B) \cdot C\right) \,, \\ h^{0,1}(\widehat{C}) &= \frac{1}{2}\left(C \cdot C - c_1(B) \cdot C\right) + 1 \;=\; g \,, \\ h^{1,1}(\widehat{C}) &= C \cdot C + 9 c_1(B) \cdot C + 2 \,, \end{aligned} \tag{3.153}$$

we can see that

$$\widehat{C}^3 = 0 \,, \qquad c_2(Y_3) \cdot \widehat{C} = 12 c_1(B) \cdot C \,, \tag{3.154}$$

and thus

$$k_3 = N c_1(B) \cdot C \,, \tag{3.155}$$

for an M5-brane wrapping any elliptic surface embedded inside an elliptic Calabi–Yau as discussed. Such a result of course also follows directly from the expression (3.141) for k_3 when one sets $M = 0$.

When the divisor is not ample we follow the idea in [97] that k_3 is really a substitute for computing the anomaly associated with the diagonal of the superconformal R-symmetry, k_r, with an additional flavour symmetry that only emerges, from the M5-brane point of view, in the IR

$$k_r = k_3 - k_F \,, \tag{3.156}$$

where k_F is the level of the emergent $SU(2)_F$ flavour symmetry.

In order to make progress in determining this flavour symmetry, we go back to the D3 brane setup in Type IIB. The reason why this is useful is that in the Type IIB side a *flavour* (*i.e.* non-R) symmetry is realised geometrically[19], simply because the normal bundle of the wrapped D3 branes is $SO(4)_T$, while the normal bundle of the wrapped M5 branes is $SO(3)_T$. Notice that while R-symmetries are ambiguous, because mixing an R-symmetry with a flavour symmetry is still an R-symmetry, flavour symmetries do not have this ambiguity.

From the self-dual string in 6d, as is discussed in section 3.5.2, we know exactly one flavour symmetry, which is the $SU(2)_L$ arising from the transverse rotations to the string, and further we can observe from the spectrum that the $SO(3)_T$ charges of the multiplets from the M5-brane on $\widehat{C}$ are the diagonal of the $SU(2)_R$ and $SU(2)_L$ charges of the multiplets from the D3-brane on C [18]. As it is the only flavour symmetry that we know is always present, and since it combines with the superconformal R-symmetry in the correct way to form $SO(3)_T$ we are justified in

[19] Although the two setups are related by a T-duality, the symmetries on the two sides do not necessarily match manifestly. This is also true at the level of isometries of supergravity solutions, although notice that presently one of the two setups does not admit a supergravity description.

conjecturing that the flavour symmetry, $SU(2)_F$, which we do not observe the origin of in the M-theory, has level, k_F, which we must subtract off to compute the k_r is none other than k_L.

From the analysis of the self-dual string we have that

$$k_L = -\frac{1}{2}N^2 C \cdot C + \frac{1}{2}Nc_1(B) \cdot C \,, \tag{3.157}$$

however, as discussed in [19], this anomaly coefficient is not quite identified with the level of the $SU(2)_L$ symmetry on the combined theory. In the anomaly coefficient of the $SU(2)_L$ anomaly there is a fictitious contribution from the centre-of-mass hypermultiplet. This universal hypermultiplet is charged under the $SU(2)_L$ however there is no $SU(2)_L$ current algebra acting on these modes. The level of the $SU(2)_L$ current algebra on the combined theory is then determined by subtracting the contribution[20] of $k_L^{\text{CoM}} = +1$ from k_L to find that the level is

$$k_L - 1 \,. \tag{3.158}$$

This is then the level of the flavour symmetry of the combined theory including the centre of mass which we then subtract from k_3, which is the level of the $SO(3)_T$ normal bundle anomaly of the combined theory, to determine the level of the superconformal R-symmetry of the combined theory.

As such the right-moving central charge as determined via the M5-brane anomaly inflow when $M = 0$ is

$$c_R = 6(k_3 - (k_L - 1)) = 3N^2 C \cdot C + 3Nc_1(B) \cdot C + 6 \,. \tag{3.159}$$

We emphasise again that, as expected, this is the central charge for the combined theory, *i.e.* the interacting theory together with the centre of mass. Further, we can observe that this identifies the superconformal R-symmetry level as

$$k_r = k_3 - (k_L - 1) = k_R \,, \tag{3.160}$$

demonstrating our statement in the previous subsection that the superconformal R-symmetry is identified with $SU(2)_R$ for all N. In this analysis we are working under the assumption that generically there is only one $SU(2)$ flavour symmetry in the IR, and that that flavour symmetry is $SU(2)_L$. If there are additional flavour symmetries then these could in principle also mix with the superconformal R-symmetry to form k_3 and these would need to be subtracted in addition.

[20]We note that there is a difference of an overall minus sign between here and [19].

3.5.4 Summary and Comparison

Let us finally summarise and compare the results of all the computations (from anomalies and holography) of central charges presented in this section. The theory to which the worldvolume theory on the string flows in the IR consists of a direct sum of two SCFTs; the generically non-trivial and the centre-of-mass conformal field theories. We shall refer to the former as the SCFT part. Depending on the method used we either compute properties of the SCFT, or else of the combined theory. Generally speaking we shall be interested in comparing the central charges of the SCFT, not including the centre of mass; these are the quantities naturally computed by the AdS duals as the centre of mass decouples in the near-horizon geometry.

The Spectrum

For a single D3-brane wrapping a curve C in the base of an elliptic threefold, or equivalently for a single M5-brane wrapping the elliptic surface $\widehat{C}$, the massless spectrum can be computed explicitly. The central charges as computed directly from the UV spectrum are

$$\text{Spectrum } (N=1): \qquad \begin{aligned} c_R &= 3C \cdot C + 3c_1(B) \cdot C + 6\,, \\ c_L &= 3C \cdot C + 9c_1(B) \cdot C + 6\,. \end{aligned} \tag{3.161}$$

These are the central charges for the combined theory, including the centre-of-mass modes. The scalar fields parametrising the position of the string in the transverse 5d or 6d space are contained inside of a single hypermultiplet, which is then referred to as the centre-of-mass hypermultiplet, and contributes to the central charges

$$(c_L^{\text{CoM}}, c_R^{\text{CoM}}) = (4, 6)\,. \tag{3.162}$$

Subtracting off these modes gives the central charges for the IR SCFT on the world-volume of the string.

Anomaly Polynomial of Self-dual Strings

In [93] the anomaly polynomial for the self-dual string in 6d was written down, as we discussed in section 3.5.2. This is the anomaly polynomial for the combined theory including both the centre-of-mass and SCFT sectors. The combined theory on the string has a global symmetry group

$$SU(2)_R \times SU(2)_L \times SU(2)_I\,, \tag{3.163}$$

where $SU(2)_R \times SU(2)_L$ comes from the transverse rotations to the string in 6d, and $SU(2)_R \times SU(2)_I$ is the UV R-symmetry of the worldvolume theory of the string. We are interested in computing from this anomaly polynomial the central charges of the SCFT in the IR. First one can determine the difference of the central charges of the combined theory from the gravitational anomaly

$$c_L - c_R = 6Nc_1(B) \cdot C\,. \tag{3.164}$$

To determine the right-moving central charge of the SCFT we need to know the level of the superconformal $SU(2)_r$ R-symmetry, which should be one of the $SU(2)$ factors inside the $SO(4)$ UV R-symmetry. Furthermore, identifying $SU(2)_R$ with the IR R-symmetry, as discussed in the previous subsection, we have computed that

$$c_R = 6k_R = 3N^2 C \cdot C + 3Nc_1(B) \cdot C\,. \tag{3.165}$$

This matches the right-moving central charge computed for the SCFT from the spectrum for $N = 1$, as expected. If we subtract the free hypermultiplet constituting the centre-of-mass degree of freedom from the difference of the right- and left-moving central charges then we can also determine the left-moving central charge for the SCFT as

$$c_L = 3N^2 C \cdot C + 9Nc_1(B) \cdot C + 2\,. \tag{3.166}$$

Again this matches the spectrum when $N = 1$ as expected.

Type IIB Supergravity

As discussed in section 3.2 we can also compute the central charges for the same setup from the Type IIB supergravity dual. As such a supergravity computation is necessarily in the near-horizon limit then the centre-of-mass modes are decoupled and we compute directly only the central charges of the SCFT. We will first consider the case *without KK-monopoles*, where in (3.55) we found that

$$c_R^{\text{IIB}} = 3N^2 C \cdot C + 3Nc_1(B) \cdot C\,, \tag{3.167}$$

which exactly matches the right-moving central charge of the theory from the spectrum and the anomaly analyses discussed previously. This would lead us to conclude that the Type IIB supergravity computation of c_R *is* in fact exact, meaning that there would be no quantum corrections, in this precise situation, as any sub-subleading correction would ruin the precise matching with the result in (3.165).

In (3.55) we also determined the left-moving central charge to be

$$c_L^{\text{IIB}} = 3N^2 C \cdot C + 9Nc_1(B) \cdot C\,, \tag{3.168}$$

where we remind the reader that this result is only expected to be accurate to order in $\mathcal{O}(N)$, and we expect from the alternate approaches to the computation of the same quantity that the full result, including quantum corrections, should have an additional $+2$.

In principle, from the Type IIB supergravity one should be able to determine the holographic central charges also for $M \geq 1$, where there are in addition M KK-monopoles in the system. However, as we discussed in section 3.2, in this case we can compute reliably only the leading order, $\mathcal{O}(N^2)$, coefficients. To determine the correct $\mathcal{O}(N)$ contributions to the anomalies we would need to incorporate the effect of the KK-monopoles.

11d Supergravity

In order for the 11d supergravity solution to exist it is necessary that the divisor wrapped by the M5-brane is an ample divisor in the Calabi–Yau threefold, and from appendix B.2.3 we can see that this generally requires that $M \geq 1$. In this section we shall take $M = 1$ principally so as to compare with the majority of the different approaches, and we will show a matching for $M \geq 1$ result at the end. For $M = 1$ (the 11d supergravity setup dual to *one* KK-monopole in Type IIB) in section 3.4 we computed the central charges to be

$$\begin{aligned} c_R^{11} &= 3N^2 C \cdot C + 3N c_1(B) \cdot C + 6\,, \\ c_L^{11} &= 3N^2 C \cdot C + 9N c_1(B) \cdot C + 2 + h^{1,1}(B)\,. \end{aligned} \tag{3.169}$$

These central charges are said to be exact in [74] as they can be determined from an anomaly analysis. Since the exactness follows from an anomaly argument these central charges should be the central charges for the full combined theory, including the centre-of-mass degrees of freedom. Given that the centre-of-mass contribution should be universal, regardless of the values of M, N, we can similarly subtract one universal hypermultiplet to determine the central charges of the IR SCFT. We notice that the leading and sub-leading terms are consistent with all other methods of computations for one or no KK-monopole. As discussed in section 3.1.3, in the near-horizon limit there is no difference between the setup with one or no KK-monopoles, and thus the leading contribution to the central charges must be identical. We find that the central charges match also at the subleading order, and in fact the expression for c_R matches the results obtained in the case without KK-monopoles exactly, but it is not clear to us whether this is accidental or not. On the other hand, it makes sense that both c_R and c_L do not both match exactly across the configurations with one or no KK-monopole, as the difference $c_L - c_R$ is a quantity that can be computed purely in the UV, and in the UV the single KK-monopole is apparent.

The general result for all $M > 0$ was given in (3.117) and reads

$$\begin{aligned} c_R^{11} &= 3N^2 M C \cdot C + 3N(2 - M^2) c_1(B) \cdot C + M^3(10 - h^{1,1}(B)) + M(h^{1,1}(B) - 4)\,, \\ c_L^{11} &= 3N^2 M C \cdot C + 3N(4 - M^2) c_1(B) \cdot C + M^3(10 - h^{1,1}(B)) + 2M(h^{1,1}(B) - 4)\,, \end{aligned} \tag{3.170}$$

but as discussed previously, we have not determined these in the Type IIB picture, beyond the leading $\mathcal{O}(N^2)$ order. At this order we indeed find perfect agreement for any N and M, see (3.58).

M5-brane Anomaly Inflow

Another M-theory approach that one can take to determine the central charges involves computing the anomaly polynomial to the string via M5-brane anomaly inflow as described in section 3.5.1. When the divisor wrapped by the M5-brane is ample in the Calabi–Yau then this approach involves effectively the same computation as was used to determine the central charges from 11d supergravity, and is also a computation of the central charges of the combined theory. The results for the central charges for $M > 0$ from the anomaly inflow are then the same as those given in (3.117) from the 11d supergravity.

The inflow computation however is valid for any divisor D even if it is not ample in the Calabi–Yau. As such, here we shall be mainly interested in the central charges for the $M = 0$ case where the M5-brane wraps simply $\widehat{C}$. As described in section 3.5.3 this approach does not directly compute the central charge, but instead computes the anomaly coefficient associated to the $SO(3)_T$ normal bundle anomaly, and the gravitational anomaly which fixes

$$c_L - c_R = 6Nc_1(B) \cdot C \,. \tag{3.171}$$

It is known that when the divisor wrapped is ample the computation of the anomaly coefficient k_3 is a suitable substitute computation for the anomaly coefficient of the superconformal R-symmetry, k_r. However when the wrapped divisor is not ample one must subtract an emergent IR flavour symmetry from k_3 to determine the superconformal R-symmetry. As discussed in section 3.5.3 we can determine the flavour symmetry which mixes with the superconformal R-symmetry and we can then compute

$$c_R = 6(k_3 - (k_L - 1)) = 3N^2 C \cdot C + 3Nc_1(B) \cdot C + 6 \,, \tag{3.172}$$

which is the central charge of the combined theory. Further one can determine the left-moving central charge of the combined theory as

$$c_L = 3N^2 C \cdot C + 9Nc_1(B) \cdot C + 6 \,. \tag{3.173}$$

3.6 Concluding Remarks

New holographic setups which allow for a controlled computational framework for both the perturbative gauge theory as well as the dual gravitational/string theory, are difficult to come by. We have studied the most general class of $(0, 4)$ AdS_3 solutions of Type IIB supergravity in the absence of three-form fluxes, which allow for a varying axio-dilaton τ, consistent with the $SL(2,\mathbb{Z})$ duality, *i.e.* F-theory solutions.

The field theory duals arise from D3-branes wrapped on curves in the base of elliptic Calabi–Yau threefold compactifications studied in [18,19]. The solutions that we have found to be the most general of this kind are of the type $\mathrm{AdS}_3 \times S_3/\Gamma \times B$, where B is the base of an elliptic Calabi–Yau threefold, and the profile of the axio-dilaton is determined in terms of the complex structure of the elliptic fiber.

Conceptually there are various points that make this duality more subtle than those involving Type IIB solutions with constant τ. First of all the profile of the axio-dilaton has to be such that τ is singular along curves in the base B. This in turn implies that the metric on the base cannot be smooth everywhere, and thus some care needs to be taken in order to reliably apply a supergravity analysis. This is in particular subtle in Type IIB as the compactification space does not include the elliptic fiber, but only the base. Key to corroborating the consistency of this solution is the duality to 11d supergravity, that we can perform for the solutions when $\Gamma = \mathbb{Z}_M$. We showed that in 11d supergravity these solutions are of the form $\mathrm{AdS}_3 \times S^2 \times Y_3$, where the elliptic Calabi–Yau threefold Y_3 can be resolved and has a smooth Ricci-flat Kähler metric.

Another class of $(0,4)$ strings in F-theory compactifications to 6d are the so-called non-Higgsable cluster strings. As we recalled earlier, these are obtained from D3-branes wrapped on collapsed curves in Calabi–Yau threefolds, which have singular algebraic varieties as base manifolds. In particular, these singularities can be thought of as arising from the collapse of a curve $C^{\mathrm{NHC}} \simeq \mathbb{P}^1$ in the local geometry of $\mathcal{O}(-n) \to \mathbb{P}^1$, where the curve has self-intersection $C^{\mathrm{NHC}} \cdot C^{\mathrm{NHC}} = -n < 0$. These can be embedded in a compact geometry by projectivizing, which results in the Hirzebruch surfaces $\mathbb{F}_n$. It is then tantalizing to speculate that our solutions might capture some features of the NHC strings by choosing the Kähler base to be $B = \mathbb{F}_n$, or their singular limits, *i.e.* the weighted projective spaces $\mathbb{P}^{(1,1,n)}$. On the other hand, since C^{NHC} is not ample, this simple setup cannot be found within the class of solutions discussed in this thesis. Our attempts to reproduce features of the NHC strings in this holographic setup have not been successful, and it remains an open problem to determine what the appropriate holographic duals of these SCFTs, if they exist, are.

In [18] a class of 2d $(0,2)$ theories were obtained, from D3-branes wrapped in the base of elliptic Calabi–Yau four- and fivefolds. These are very closely related setups to the ones studied here, and naturally finding AdS_3 duals to these 2d SCFTs would be very interesting. In relation to the solutions found here, the case of Calabi–Yau fivefolds is closely related to our F-theory solutions with KK-monopole. The F-theory compactification space is $Y_3 \times TN_M$, which is a special Calabi–Yau fivefold. F-theory on elliptic Calabi–Yau fivefolds has only recently been investigated in [98,99] and result in 2d (0,2) theories for generic Calabi–Yau fivefolds. In view of this, it would be interesting to study our AdS_3 solutions with $\Gamma = \mathbb{Z}_M$ in relation to the near horizon limits of D3-branes in Calabi–Yau five-fold compactifications of the type $Y_3 \times TN_M$ and determine the spectrum for general M as in [18].

Chapter 4

$(0,2)$ solutions and field theory duals

Having discussed solutions with $(0,4)$ supersymmetry we now turn our attention to the richer class of theories with $(0,2)$ supersymmetry. Recall that the supergravity solution is determined by the choice of Kähler base satisfying the master equation (2.36). In practice we shall instead look for solutions to the F-theoretic reformulation, (2.59) and solve for an elliptically fibered Kähler four-fold $\mathcal{Y}_8^\tau$. We will study two new classes of solutions, which result from different specialisations of the Kähler four-fold.

4.1 New $\mathcal{N}=(0,2)$ Solutions with Varying τ

The first type of solution is a specialisation of the F-theoretic reformulation in section 2.2.5, where $\widetilde{M}_6$ is a direct product of a complex curve and a complex surface,

$$\widetilde{M}_6 = \Sigma \times \mathcal{M}_4\,, \tag{4.1}$$

such that the elliptic fibration is only non-trivial over one of these subspaces. There are two cases

$$\begin{aligned} \text{Elliptic Surface:} \qquad & \mathcal{Y}_8^\tau = (\mathbb{E}_\tau \to \Sigma) \times \mathcal{M}_4 = \mathcal{S}_4^\tau \times \mathcal{M}_4 \\ \text{Elliptic Three-fold:} \qquad & \mathcal{Y}_8^\tau = \Sigma \times (\mathbb{E}_\tau \to \mathcal{M}_4) = \Sigma \times \mathcal{T}_6^\tau\,. \end{aligned} \tag{4.2}$$

where none of the factors has a Ricci-flat metric. This class will correspond in the dual field theory to "universal twist solutions", which generalise to varying τ the universal twist solutions in [58], that were originally found in [100]. We will see that they are dimensional reductions with topological duality twist of 4d $\mathcal{N}=1$ SCFTs with rational R charges, with varying coupling. In this class of solutions we do not assume that the elliptic fibration over Σ or $\mathcal{M}_4$ are Ricci-flat. In fact the "master equation" implies that they are not. These solutions will be studied in section 4.1.1.

Another class of solutions can be obtained by a similar splitting, where in addition we now require that the factor with the non-trivial elliptic fibration is Ricci-flat, i.e. has a Calabi-Yau $(4-s)$-fold factor $\mathcal{Y}^\tau_{2(4-s)}$

$$\mathcal{Y}^\tau_8 = \mathcal{Y}^\tau_{2(4-s)} \times \mathcal{M}_{2s} \,. \tag{4.3}$$

Clearly $\mathcal{M}_{2s}$ has to be Kähler as well and only the values $s = 1, 2$ are interesting[1]. Inserting the direct product metric into (2.59) one immediately finds that the Kähler metric on $\mathcal{M}_{2s}$ must again obey the same equation originally found in [55], namely

$$\Box_{\mathcal{M}_{2s}} R^{(\mathcal{M}_{2s})} - \frac{1}{2} R^{(\mathcal{M}_{2s})2} + R^{(\mathcal{M}_{2s})}_{ij} R^{(\mathcal{M}_{2s})ij} = 0 \ . \tag{4.4}$$

We shall first consider the case when $s = 1$ where $\mathcal{Y}^\tau_8$ is the direct product of an elliptically fibered Calabi–Yau three-fold and a Riemann surface before considering the $s = 2$ case. As we shall show the former recovers the $(0, 4)$ solutions determined in [59] whilst the latter gives rise to a new class of strictly $(0, 2)$ supersymmetic solutions. These solutions will be the subject of section 4.1.2.

4.1.1 Universal Twist Solutions

In this section we begin with the product ansatz in (4.1)

$$\mathrm{d}s^2(\widetilde{\mathcal{M}}_6) = \mathrm{d}s^2(\Sigma) + \mathrm{d}s^2(\mathcal{M}_4) \,, \tag{4.5}$$

where Σ is a complex curve and $\mathcal{M}_4$ a Kähler surface. It is most convenient to express our ansatz in the reformulation of section 2.2.5. The Ricci-form of the 8d space $\mathcal{Y}^\tau_8$, which is the elliptic fibration over $\widetilde{\mathcal{M}}_6$ is taken to be

$$\mathfrak{R}_{\mathcal{Y}} = k_1 J_{\mathcal{M}_4} + k_2 J_\Sigma \,, \tag{4.6}$$

where k_1 and k_2 are constants. We consider the two cases outlined in (4.2): τ varies non-trivially only over the curve Σ giving an elliptic surface, or τ varies non-trivially only over $\mathcal{M}_4$ giving an elliptic three-fold. Though the supergravity solutions are distinct much of the analysis will be similar, and therefore it will be useful to keep the discussion as general as possible. Inserting the above ansatz into the 'master equation' (2.59) the necessary condition is

$$k_1(k_1 + 2k_2) = 0 \qquad \text{and} \qquad R_{\mathcal{Y}} = 4k_1 + 2k_2 > 0 \ . \tag{4.7}$$

[1]Note that $s = 0$ is ruled out because the Ricci scalar of $\mathcal{Y}^\tau_8$, and therefore also the warp factor e^{-4H}, vanishes in this case. $s = 3$ corresponds to $\mathcal{Y}_8 = T^2 \times \mathcal{M}_6$, which has constant axio-dilaton [55].

Clearly to solve (4.7) either $k_1 = 0$ or $k_1 = -2k_2$. The former recovers the $(0,4)$ solution discussed in [59][2]. We therefore consider the latter solution in the remainder of this section. Evaluated on such a solution the Ricci scalar is $R_{\mathcal{Y}} = -6k_2$ and thus the positivity constraint of the Ricci scalar implies that $k_2 < 0$. The overall scale of the Kähler metric on $\widetilde{\mathcal{M}}_6$ may be removed by a coordinate change, thus without loss of generality we may set k_2 to be any negative value we wish, for convenience we choose $k_2 = -3$. The 10d solution in Einstein frame is

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{2}{3}\left(\mathrm{d}s^2(\mathrm{AdS}_3) + \frac{9}{4}\left(\frac{1}{9}(\mathrm{d}\chi+\rho)^2 + \mathrm{d}s^2(\mathcal{M}_4) + \mathrm{d}s^2(\Sigma)\right)\right) \ , && (4.8)\\
\mathrm{e}^{-4H} &= \frac{9}{4} \ , && (4.9)\\
F^{(2)} &= -\frac{2}{3}(4J_\Sigma + J_{\mathcal{M}_4}) \ , && (4.10)\\
F &= -\frac{3}{4}(\mathrm{d}\chi+\rho)\wedge J_{\mathcal{M}_4}\wedge(2J_{\mathcal{M}_4}+J_\Sigma) - \frac{2}{3}\mathrm{dvol}(\mathrm{AdS}_3)\wedge(4J_\Sigma + J_{\mathcal{M}_4}) \ , && (4.11)\\
\rho &= -6\mathcal{A}_{\mathcal{M}_4} + 3\mathcal{A}_\Sigma \ , && (4.12)\\
\mathrm{d}\mathcal{A}_i &= J_i \ . && (4.13)
\end{aligned}$$

The Ricci form on $\mathcal{Y}_8^\tau$ becomes $\mathfrak{R}_{\mathcal{Y}} = 6J_{\mathcal{M}_4} - 3J_\Sigma$, in matrix block form this is

$$\mathfrak{R}_{\mathcal{Y}} = \left(\begin{array}{c|c|c} -3J_\Sigma & & \\ \hline & 0 & \\ \hline & & 6J_{\mathcal{M}_4} \end{array}\right) \ . \qquad (4.14)$$

In the above we have not specified over which factor in $\widetilde{\mathcal{M}}_6$, τ varies non-trivially. In the following we shall consider the two cases in which τ varies non-trivially only over Σ, giving an elliptic surface, or over $\mathcal{M}_4$, giving an elliptic three-fold. Before proceeding we emphasise that we are not aware of any existence results for metrics on either the elliptic surface or the elliptic three-fold with the specific conditions imposed on the curvatures (in particular let us re-emphasise that these are not Ricci-flat). We will assume that such metrics exist on these spaces with the Ricci-form given as above. It would indeed be of great interest to develop the mathematics that shows the existence of such metrics. Of course the bases of these elliptic fibrations will have singularities at points where τ becomes singular, but by assumption they will be otherwise smooth. In the following sections we analyse the two distinct types of solutions discussed above. The consistency of the holographic computations using these solutions with the proposed field theory duals corroborates our conjecture that these metrics exist.

[2] We will recover these (0,4) solutions in a slightly different way in section 4.1.2.

Elliptic Surface Case

Let us first consider the case where τ varies non-trivially only over Σ. We require the metric on $\mathcal{Y}_8^\tau$ to factorise as

$$\mathrm{d}s^2(\mathcal{Y}_8^\tau) = \mathrm{d}s^2(\mathcal{S}_4^\tau) + \mathrm{d}s^2(\mathcal{M}_4)\,, \tag{4.15}$$

where $\mathbb{E}_\tau \hookrightarrow \mathcal{S}_4^\tau \to \Sigma$ is an elliptic surface with section, over Σ. The Ricci curvature then factorises into two 4×4 blocks, and (4.14) reads

$$\left(\begin{array}{c|c} \mathfrak{R}_{\mathcal{S}_4^\tau} & \\ \hline & \mathfrak{R}_{\mathcal{M}_4} \end{array}\right) = \left(\begin{array}{c|c|c} -3J_\Sigma & & \\ \hline & 0 & \\ \hline & & 6J_{\mathcal{M}_4} \end{array}\right) . \tag{4.16}$$

To solve this equation we therefore have that the metric on $\mathcal{M}_4$ is Kähler–Einstein with Ricci-form $\mathfrak{R}_{\mathcal{M}_4} = 6J_{\mathcal{M}_4}$, and we require the existence of a metric on the elliptic surface $\mathcal{S}_4^\tau$ to satisfy

$$\mathfrak{R}_{\mathcal{S}_4^\tau} = -3J_\Sigma \quad \Longleftrightarrow \quad \mathfrak{R}_\Sigma + \mathrm{d}Q = -3J_\Sigma\ . \tag{4.17}$$

Notice that the Kähler–Einstein metric on $\mathcal{M}_4$ has the normalisation of the base of a Sasaki–Einstein manifold. In fact the one form dual to the Reeb vector field of the Sasaki–Einstein manifold is given by $-\frac{1}{3}(\mathrm{d}\chi + \rho)$ at fixed coordinate on Σ. We conclude that at fixed coordinate on Σ the $U(1)$ fibration over $\mathcal{M}_4$ is a (quasi-regular) Sasaki–Einstein manifold.

Solutions of this form, where Σ is the constant curvature Riemann surface $\mathbb{H}^2$ have been studied in [100], however there are some differences once τ is allowed to vary non-trivially over Σ. Topologically the 7d internal space is a $U(1)$ fibration over $\mathcal{M}_4 \times \Sigma$. Such fibrations are well-defined if the first Chern class of the bundle is integral over all two-cycles in $H_2(\mathcal{M}_4 \times \Sigma, \mathbb{Z})$. Let the period of χ be $2\pi\ell$, then we require

$$\frac{1}{2\pi}\frac{1}{\ell}\mathrm{d}\rho = \frac{1}{2\pi\ell}(-6J_{\mathcal{M}_4} + 3J_\Sigma) \in H^2(\mathcal{M}_4 \times \Sigma, \mathbb{Z})\ . \tag{4.18}$$

This may be rephrased in terms of the elliptic surface $\mathcal{S}_4^\tau$ with base Σ as

$$c_1(U(1)) = -\frac{1}{\ell}(c_1(\mathcal{M}_4) + c_1(\mathcal{S}_4^\tau)|_\Sigma) \in H^2(\mathcal{M}_4 \times \Sigma, \mathbb{Z})\ . \tag{4.19}$$

Notice that the non-trivial elliptic fibration implies that the quantisation condition differs to that in [100]. Concretely we have used the first Chern class of the elliptic surface $\mathcal{S}_4^\tau$ to rewrite the condition on $c_1(U(1))$. A convenient basis for $H_2(\widetilde{\mathcal{M}}_6)$ is furnished by the set $\{\Sigma, \Sigma_\alpha\}$ where $\{\Sigma_\alpha\}$ is a basis of $H_2(\mathcal{M}_4, \mathbb{Z})$. Then $c_1(U(1))$

being integer implies

$$\frac{1}{2\pi}\int_{\Sigma} c_1(U(1)) = -\frac{1}{\ell}(2(g-1)+\deg(\mathcal{L}_D)) \in \mathbb{Z}$$
$$\frac{1}{2\pi}\int_{\Sigma_a} c_1(U(1)) = -\frac{\tilde{m}n_\alpha}{\ell} \in \mathbb{Z}\,, \tag{4.20}$$

where $\tilde{m}$ is the Fano-index of $\mathcal{M}_4$, see appendix B of [100] for a review of properties of 4d Kähler–Einstein spaces, and n_α are relatively prime integers. The period ℓ of χ must be a divisor of both $\tilde{m}$ and $(2(g-1)+\deg(\mathcal{L}_D))$ and consequently it has maximal value $\ell = \gcd\{\tilde{m}, (2(g-1)+\deg(\mathcal{L}_D))\}$. Recall that this construction only works for the regular and quasi-regular Sasaki–Einstein metrics [100][3].

Flux Quantisation

The cycles of interest are the compact five-cycles of the geometry, of which there are two classes. The first is the five-cycle given at fixed Σ coordinates, which is a Sasaki–Einstein (SE) manifold. The second class of five-cycles, which we denote D_α, are obtained as $U(1)$ fibrations over $\Sigma_\alpha \times \Sigma$, where $\Sigma_\alpha \in H_2(\mathcal{M}_4,\mathbb{Z})$. For the former we find

$$N(\mathrm{SE}_5) = \frac{1}{(2\pi\ell_\mathrm{s})^4 g_\mathrm{s}}\int_{SE_5} F = \frac{9}{(2\pi\ell_\mathrm{s} m)^4 g_\mathrm{s}}\mathrm{vol}(SE_5)\ , \tag{4.21}$$

where the volumes are computed with the canonical Sasaki–Einstein metrics, which have Ricci-tensor satisfying $R_{\mu\nu} = 4g_{\mu\nu}$. As it is necessary for the fibration to be quasi-regular we may rewrite this quantisation condition as

$$N(SE_5) = \frac{\ell M}{2^4 3\pi m^4 {\ell_\mathrm{s}}^4 g_\mathrm{s}} \in \mathbb{Z}\ , \tag{4.22}$$

where the integer M is the topological invariant

$$M = \int_{\mathcal{M}_4} c_1(\mathcal{M}_4)\wedge c_1(\mathcal{M}_4)\ . \tag{4.23}$$

For the five-cycles D_α the condition is

$$N(D_\alpha) = -\frac{\ell\tilde{m}n_\alpha(2(g-1)+\deg(\mathcal{L}_D))}{2^4 3\pi m^4 {\ell_\mathrm{s}}^4 g_\mathrm{s}} \in \mathbb{Z}\ . \tag{4.24}$$

[3]Every Sasakian manifold admits a canonically defined Killing vector field called the Reeb vector. Sasakian manifolds may be classified according to the global properties of said Reeb vector. First consider the case when the orbits of the Reeb vector are all closed and thus circles. As the Reeb is nowhere-vanishing the isotropy group is necessarily finite at every point. If the $U(1)$ action is in fact free (globally the isotropy group consists of just the identity element) then the Sasakian structure is said to be *regular*. If on the other hand the $U(1)$ action is not free everywhere it is called *quasi-regular*. Instead if the orbits of the Reeb do not all close then the Sasakian structure is said to be *irregular*. For $Y^{p,q}$ the Sasakian structure is quasi-regular when $4p^2-3q^2$ is a square, and irregular otherwise. Observe that this corresponds to the R-charges being rational or irrational.

Quantisation of the flux such that the above integers are minimal implies

$$n = \frac{\tilde{m}\ell h}{2^4 3\pi m^4 {\ell_{\mathrm{s}}}^4 g_{\mathrm{s}}}, \qquad h = \gcd\left(\frac{M}{\tilde{m}}, 2(g-1) + \deg(\mathcal{L}_D)\right) , \tag{4.25}$$

from which we obtain

$$N(SE_5) \equiv N = \frac{nM}{\tilde{m}h} , \quad N(D_\alpha) = \frac{nn_\alpha}{h}(2(g-1) + \deg(\mathcal{L}_D)) . \tag{4.26}$$

In comparing with the field theory results we shall identify the integer N as the number of D3-branes in the setup. Notice that the above analysis is a generalisation to that performed in [100], corresponding to $\deg(\mathcal{L}_D) = 0$.

Elliptic Three-fold Case

Consider now the case where τ varies non-trivially only over $\mathcal{M}_4$, so that the metric on $\mathcal{Y}_8^\tau$ factorises as

$$\mathrm{d}s^2(\mathcal{Y}_8^\tau) = \mathrm{d}s^2(\Sigma) + \mathrm{d}s^2(\mathcal{T}_6^\tau) , \tag{4.27}$$

where $\mathbb{E}_\tau \hookrightarrow \mathcal{T}_6^\tau \to \mathcal{M}_4$ is the elliptic three-fold. The Ricci curvature of this metric now factorises in one 2×2 block and one 6×6 block, and (4.14) reads

$$\left(\begin{array}{c|c} \mathfrak{R}_\Sigma & \\ \hline & \mathfrak{R}_{\mathcal{T}_6^\tau} \end{array}\right) = \left(\begin{array}{c|c|c} -3J_\Sigma & & \\ \hline & 0 & \\ \hline & & 6J_{\mathcal{M}_4} \end{array}\right) . \tag{4.28}$$

The upper block of this equation implies that the metric on the Riemann surface has constant curvature $\mathfrak{R}_\Sigma = -3J_\Sigma$. We then require the existence of a metric on the elliptic three-fold $\mathcal{T}_6^\tau$ to satisfy

$$\mathfrak{R}_{\mathcal{T}_6^\tau} = 6J_{\mathcal{M}_4} \quad \Longleftrightarrow \quad \mathfrak{R}_{\mathcal{M}_4} + \mathrm{d}Q = 6J_{\mathcal{M}_4} . \tag{4.29}$$

In fact, the elliptic three-fold $\mathcal{T}_6^\tau$ is precisely that appeared in section 2.4.3. At fixed coordinates on Σ, the solutions can be obtained in the same way as the AdS_5 solutions discussed in section 2.4.3. We nevertheless give a brief discussion on global properties of the solutions following the above. Topologically the solution is again a $U(1)$ fibration over a Kähler base. Giving χ period $2\pi\ell$ as before the first Chern class of the $U(1)$ bundle is

$$c_1(U(1)) = -\frac{1}{\ell}(c_1(\Sigma) + c_1(\mathcal{T}_6^\tau)|_{\mathcal{M}_4}) . \tag{4.30}$$

Using the same basis as previously we require

$$\frac{1}{2\pi}\int_{\Sigma_\alpha} c_1(U(1)) = \frac{1}{\ell}(c_1(\mathcal{T}_6^\tau)\cdot\Sigma_\alpha) \in \mathbb{Z}\ , \quad \frac{1}{2\pi}\int_{\Sigma} c_1(U(1)) = \frac{\chi(\Sigma)}{\ell} \in \mathbb{Z}\ . \tag{4.31}$$

Here $\chi(\Sigma)$ is the Euler number of the Riemann surface Σ. The period ℓ must divide both $\chi(\Sigma)$ and $c_1(\mathcal{T}_6^\tau)\cdot\Sigma_\alpha$ for all α.

Flux Quantisation

Recall that at fixed coordinates on the constant curvature Riemann surface Σ, the metric is no longer Einstein, though it remains Sasakian. We will refer to this space as $\mathcal{M}_5^\tau$ as it will be related to the $\mathcal{M}_5^\tau$ of section 2.4.3. The possible five-cycles are as before and we keep the same notation as in the previous quantisation condition. Then the quantisation condition is

$$N(\mathcal{M}_5^\tau) = \frac{1}{(2\pi\ell_s)^4 g_s}\int_{\mathcal{M}_5^\tau} F = \frac{9}{(2\pi m\ell_s)^4 g_s}\mathrm{vol}(\mathcal{M}_5^\tau)\,, \tag{4.32}$$

which has the same form as for the first class of solutions. We may rewrite the volume of $\mathcal{M}_5^\tau$ as

$$\mathrm{vol}(\mathcal{M}_5^\tau) = \frac{1}{2}\int_{\mathcal{M}_5^\tau}\frac{1}{3}\mathrm{d}\chi\wedge J_{\mathcal{M}_4}\wedge J_{\mathcal{M}_4} = \frac{\pi^3\ell}{27}\int_{\mathcal{M}_4}\left(c_1(\mathcal{M}_4)^2 - 2c_1(\mathcal{M}_4)c_1(\mathcal{L}_D) + c_1(\mathcal{L}_D)^2\right)\ , \tag{4.33}$$

where the integral on the right-hand side is an integer, given by the sum of three topological numbers, whose value we denote by $\widetilde{M}$. Then

$$N(\mathcal{M}_5^\tau) = \frac{\ell\widetilde{M}}{2^4\cdot 3\pi{\ell_s}^4 m^4 g_s}\ . \tag{4.34}$$

The quantisation over the remaining five-cycles gives

$$N(D_\alpha) = \frac{\chi(\Sigma)\ell}{2^4\cdot 3\pi m^4{\ell_s}^4 g_s}\left(\tilde{m}n_\alpha - \int_{\Sigma_\alpha} c_1(\mathcal{L}_D)\right)\ . \tag{4.35}$$

As before we impose that the fluxes are minimal integers through all integral cycles which implies the quantisation of the length scale m as

$$n = \frac{\ell h}{2^4\cdot 3\pi m^4{\ell_s}^4 g_s}\,,\qquad h = \gcd\left[\tilde{M}, \chi(\Sigma)gcd\left(\left\{\tilde{m}n_\alpha - \int_{\Sigma_\alpha} c_1(\mathcal{L}_D)\right\}_\alpha\right)\right]\ . \tag{4.36}$$

We have

$$N(\mathcal{M}_5^\tau) \equiv N = \frac{\widetilde{M}n}{h}\ , \quad N(D_\alpha) = \frac{\chi(\Sigma)n}{h}\left(\tilde{m}n_\alpha - \int_{\Sigma_\alpha} c_1(\mathcal{L}_D)\right)\ . \tag{4.37}$$

4.1.2 Solutions with Calabi-Yau Factors

We now consider the ansatz (4.3), with one of the factors in $\mathcal{Y}_8^\tau$ an elliptically fibered Calabi-Yau.

Recovering the $(0,4)$ Solutions

The case $s=1$, i.e. $\mathcal{Y}_8^\tau = \mathcal{Y}_6 \times \Sigma$, where $\mathcal{Y}_6$ is an elliptically fibered Calabi–Yau three-fold and Σ is a complex curve recovers the classification of $\mathcal{N}=(0,4)$ theories that we presented in [59]. The metric is

$$\mathrm{d}s^2(\mathcal{Y}_8^\tau) = \mathrm{d}s^2(\mathcal{Y}_6) + \mathrm{d}s^2(\Sigma) \ , \tag{4.38}$$

and as any Riemann surface is conformally flat we may write the metric on $\mathcal{M}_2$ as

$$\mathrm{d}s^2(\Sigma) = \mathrm{e}^{-2f(x,y)}(\mathrm{d}x^2 + \mathrm{d}y^2) \ . \tag{4.39}$$

A Riemann surface trivially satisfies $R^2 = 2R_{\mu\nu}R^{\mu\nu}$ and therefore (4.4) reduces to

$$\Box_\Sigma R^\Sigma = 0 \ . \tag{4.40}$$

On any smooth compact manifold any bounded harmonic function is constant and it follows that for a smooth and compact internal manifold we must have that R^Σ is constant and therefore the Riemann surface is of constant curvature[4]. For positive curvature, as is necessary by (2.60), the only possibility is a round two-sphere and it follows that the only solutions are of the form

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathrm{AdS}_3) + \mathrm{d}s^2(S^3/\Gamma) + \mathrm{d}s^2(B_4) \tag{4.41}$$

where B_4 is the base of $\mathcal{Y}_6$, the elliptically fibered Calabi–Yau introduced above. This precisely reproduces the solutions discussed in [59] and in section 3.

Baryonic Twist Solutions

A new class of solutions with exactly $(0,2)$ supersymmetry can be obtained for $s=2$ in the ansatz (4.3), i.e. where the geometry consists of an elliptic K3 surface $\mathcal{Y}_4$ and a local Kähler surface $\mathcal{M}_4$ as factors

$$\mathrm{d}s^2(\mathcal{Y}_8^\tau) = \mathrm{d}s^2(\mathcal{Y}_4) + \mathrm{d}s^2(\mathcal{M}_4) \ . \tag{4.42}$$

[4] Removing the smoothness assumption, there could exist further (0,2) solutions where Σ has singularities. In [59] we did not make any global assumptions and therefore those are indeed the most general solutions preserving $(0,4)$ supersymmetry.

Any solution to the "master equation" (4.4) for the metric on $\mathcal{M}_4$ will furnish a solution with varying axio-dilaton. In fact, solutions have been found previously in the literature for $\mathcal{M}_4$, [56,57] and in the following section we shall discuss a particular example. We begin by writing the full local solution with varying axio-dilaton, and subsequently we will investigate its global regularity, including quantisation of the fluxes. The computations are very similar to those presented in [56, 57] for the solutions with constant axio-dilaton. We include the details below and in appendix C.1 in order to be self-contained and to highlight some subtle features, which were not emphasised before.

The solutions bear an uncanny resemblance to the five-dimensional $Y^{p,q}$ Sasaki–Einstein manifolds [32]. Following the ideas in [58], this connection will be sharpened by a dual field theory discussion in section 4.3, where we will propose that the dual 2d SCFTs are obtained from a particular twisted compactification of the $Y^{p,q}$ theories on a curve, with a varying coupling.

The local metric in string frame is

$$\begin{aligned} \mathrm{d}s^2_{IIB(SF)} = \frac{1}{\sqrt{ax\tau_2}}\Bigg[\mathrm{d}s^2(\mathrm{AdS}_3) + \frac{1}{4m^2}\Bigg(& w[\mathrm{d}\psi + g(x)D\phi]^2 + 4ax\mathrm{d}s^2(B_2) \qquad (4.43)\\ & +a\left(\frac{\mathrm{d}x^2}{x^2U} + \frac{U}{w}D\phi^2 + \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\chi^2\right)\Bigg)\Bigg] \, , \qquad (4.44) \end{aligned}$$

with RR five-form flux

$$\begin{aligned} F = & -\frac{1}{m}\mathrm{dvol}(\mathrm{AdS}_3)\wedge\left(\frac{1}{2ax^2}(D\psi - g(x)D\phi)\wedge\mathrm{d}x + 2\mathrm{dvol}(B_2) + \frac{1}{2}\mathrm{dvol}(S^2)\right)\\ & +\frac{a}{4m^4}D\psi\wedge D\phi\wedge\mathrm{d}x\wedge\left(\mathrm{dvol}(B_2) + \frac{1}{4x^2}\mathrm{dvol}(S^2)\right)\\ & +\frac{a}{4m^4}\mathrm{dvol}(B_2)\wedge\mathrm{dvol}(S^2)\wedge\left(xD\psi - \frac{U(x)}{w(x)}D\phi\right) \, . \end{aligned} \qquad (4.45)$$

The axio-dilaton varies holomorphically over $B_2 = \mathbb{P}^1$, such that the total space of the elliptic fibration $\mathcal{Y}_4$, $\mathbb{E}_\tau \hookrightarrow \mathcal{Y}_4 \to B_2$, where the axio-dilaton parametrises the complex structure of the fiber, is a K3 surface. The warp factor is $\mathrm{e}^{-4H(x)} = ax$ and in the above expressions we have used the following definitions

$$\begin{aligned} & U(x) = 1 - a(1-x)^2 \, , \quad w(x) = 1 + a(2x-1) \, , \quad g(x) = -\frac{ax}{w(x)} \, , \qquad (4.46)\\ & D\phi = \mathrm{d}\phi + \cos\theta\mathrm{d}\chi \, , \quad D\psi = \mathrm{d}\psi + g(x)D\phi \, , \end{aligned}$$

with a an integration constant. After performing the global regularity analysis, that we include in appendix C.1, one discovers that a takes rational values, given in terms of two integers $\mathfrak{p}, \mathfrak{q}$. The resulting Type IIB solution takes the form

$$\mathrm{AdS}_3 \times \mathbb{P}^1 \times \mathfrak{Y}^{\mathfrak{p},\mathfrak{q}} \, , \qquad \mathfrak{Y}^{\mathfrak{p},\mathfrak{q}} = (S^1 \to \mathbb{F}_0) \, , \qquad (4.47)$$

where $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ is a circle fibration over $\mathbb{F}_0 = S^2 \times S^2$, with Chern numbers $\mathfrak{p}$ and $\mathfrak{q}$, respectively, that are related to the parameter a as

$$a = \frac{\mathfrak{q}^2}{\mathfrak{p}^2} \, . \tag{4.48}$$

Of course the Kähler metric on this $\mathbb{F}_0$ is not the Einstein, direct-product metric on $S^2 \times S^2$.

Regularity of the metric requires that $a > 1$, which implies that the integers $\mathfrak{p}, \mathfrak{q}$ obey

$$0 < \mathfrak{p} < \mathfrak{q} \, . \tag{4.49}$$

This notation is closely related to the one in [32], and a further discussion of the relation to the standard $Y^{p,q}$ is provided in appendix C.1.

Flux Quantisation

Finally, we need to check that the flux of the solution is properly quantized, i.e.

$$N(D) = \frac{1}{(2\pi \ell_s)^4 g_s} \int_D F \in \mathbb{Z} \tag{4.50}$$

for any five-cycle $D \in H_5(\mathcal{M}_7; \mathbb{Z})$. There are two independent five-cycles in $\mathcal{M}_7 = \mathbb{P}^1 \times \mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$, namely $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ at a point on the base B_2 of the elliptic K3, and $E \times B_2 = E \times \mathbb{P}^1$, where the E is the unique generator of $H_3(\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}; \mathbb{Z})$. The flux as given in (C.4) is

$$mF^{(2)} = -\left(\frac{1}{ax^2} (D\alpha - g(x) D\phi) \wedge \mathrm{d}x + 2J_{B_2} + \frac{1}{2} \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\chi \right) \, . \tag{4.51}$$

Due to the self-duality of the five-form flux, it is the Hodge star of the above two-form that needs to be quantised. An explicit computation reveals that

$$\begin{aligned} m^4 *_7 F^{(2)} =& \frac{a}{4} D\alpha \wedge D\phi \wedge \mathrm{d}x \wedge \left(\mathrm{dvol}(B_2) + \frac{1}{4x^2} \mathrm{dvol}(S^2) \right) \\ &+ \frac{a}{4} \mathrm{dvol}(B_2) \wedge \mathrm{dvol}(S^2) \wedge \left(x D\alpha - \frac{U(x)}{w(x)} D\phi \right) \, . \end{aligned} \tag{4.52}$$

The flux through the cycles $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ is

$$\begin{aligned} \int_{\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}} *_7 F^{(2)} =& \frac{1}{m^4} \int_{\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}} \frac{a}{16x^2} \sin\theta \mathrm{d}x \wedge \mathrm{d}\theta \wedge \mathrm{d}\chi \wedge D\alpha \wedge D\phi \\ =& -\frac{(2\pi)^3}{2m^4} \left(\frac{\mathfrak{q}^4}{\mathfrak{p}(\mathfrak{p}^2 - \mathfrak{q}^2)^2} \right) \, . \end{aligned} \tag{4.53}$$

Which implies the quantisation condition

$$\frac{1}{(2\pi\ell_{\rm s})^4 g_s m^4} = \frac{N}{4\pi^3}\frac{\mathfrak{p}(\mathfrak{p}^2-\mathfrak{q}^2)^2}{\mathfrak{q}^4}\ , \tag{4.54}$$

where[5]

$$\frac{1}{(2\pi\ell_{\rm s})^4 g_s}\int_{\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}} F = -N\ , \qquad N\in\mathbb{N}\ . \tag{4.55}$$

The integer N is interpreted as the number of D3-branes along $\mathbb{R}^{1,1}\times\mathbb{P}^1$.

To perform the quantisation over the other five-cycle, we must first identify the correct generator for $H_3(\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}};\mathbb{Z})$. It is not simple to identify this three-cycle in the metric as it is not a product metric. There are four easily identifiable three-cycles at each of the degeneration surfaces with further discussion of these degeneration surfaces is provided in appendix C.1.3. Let the generator of $H_3(\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}};\mathbb{Z})$ be denoted E, and the three-cycles at each of the degeneration surfaces be E^a where $a\in\{+,-,0,\pi\}$. The closed three-form dual to the generator E is

$$\omega_3 = \frac{\mathfrak{p}^2-\mathfrak{q}^2}{(4\pi)^2}\left[D\alpha\wedge D\phi\wedge \mathrm{d}x + \left(xD\alpha - \frac{U(x)}{w(x)}D\phi\right)\wedge \mathrm{dvol}(S^2)\right] \tag{4.56}$$

and satisfies

$$\int_E \omega_3 = 1\ . \tag{4.57}$$

One may use the above three-form to verify that the following homology relations

$$E^+ = (\mathfrak{p}+\mathfrak{q})E\ ,\quad E^- = (\mathfrak{p}-\mathfrak{q})E\ ,\quad E^0 = E^\pi = -\mathfrak{p}E\ , \tag{4.58}$$

hold true. Then the integration of the five-form flux over the five-cycle $E\times B_2$ gives

$$\begin{aligned}\int_{E\times B_2} *_7 F^{(2)} &= \frac{1}{m^4}\int_{E\times B_2}\frac{a}{4}\mathrm{dvol}(B_2)\wedge\left(D\alpha\wedge D\phi\wedge\mathrm{d}x + \left(xD\alpha - \frac{U(x)}{w(x)}D\phi\right)\wedge\mathrm{dvol}(S^2)\right)\\ &= \frac{4a\pi^2}{(\mathfrak{p}^2-\mathfrak{q}^2)m^4}\int_{E\times B_2}\mathrm{dvol}(B_2)\wedge\omega_3 = \frac{4\pi^2}{m^4}\frac{\mathfrak{q}^2}{\mathfrak{p}^2(\mathfrak{p}^2-\mathfrak{q}^2)}\mathrm{vol}(B_2)\ . \end{aligned}\tag{4.59}$$

Flux quantisation imposes

$$\frac{1}{(2\pi\ell_{\rm s})^4 g_s}\int_{E\times B_2} F = -M\ , \qquad M\in\mathbb{N}\ , \tag{4.60}$$

which may be interpreted as quantisation of the volume of B_2

$$\mathrm{vol}(B_2) = \frac{M\pi}{N}\frac{\mathfrak{p}\mathfrak{q}^2}{\mathfrak{q}^2-\mathfrak{p}^2}\ . \tag{4.61}$$

This concludes the discussion of the new AdS_3 solutions in F-theory dual to $(0,2)$

[5]We chose this sign to ensure that $N>0$.

SCFTs. In the following we will use these to test the duality by comparing holographic charges with the dual field theory observables.

4.2 Holographic Charges

To compare physical observables with the dual SCFTs, we now turn to computing holographically the central charge as well as the R-charges and baryonic charges of baryonic operators, which will be compared to the dual field theories in section 4.3. At leading order in N, the results of the holographic computations presented in this section also apply, with minor modifications, to the holographic duals with constant axio-dilaton [58].

4.2.1 General Considerations

The leading order central charge is computed using the standard Brown-Henneaux prescription [72], relating it to Newton's constant G_N in 3d as

$$c_{\text{sugra}} = \frac{3}{2mG_N^{(3)}} \ . \tag{4.62}$$

This can be extracted from the solution by computing the volume of the compact part of the spacetime $\mathcal{M}_7$. We remark that in all the solutions presented above the bases of the elliptic surfaces and three-folds considered above are singular. Nevertheless, the volumes of these spaces can be computed indirectly either by using flux quantisation or relating it to various topological quantities. Here we furthermore assume that the fibration is a smooth Weierstrass model, i.e. with only I_1 fibers. This will allow us to circumnavigate having to resolve any additional singularities, in passing to an M-theory picture. A similar logic was employed in [59], and cross-checked against a smooth M-theory dual, field theory and anomalies. Using the conventions in appendix B.6 we have

$$c_{\text{sugra}} = \frac{3}{2mG_N^{(10)}} \int_{\mathcal{M}_7} \mathrm{e}^H \mathrm{dvol}(\mathcal{M}_7) \ , \tag{4.63}$$

where $G_N^{(10)} = 2^3\pi^6 {\ell_{\text{s}}}^8$ is the 10d Newton's constant.

The subleading contribution to the central charge can be determined by anomaly inflow on the 7-branes as in [59], which follows an argument presented in [37]. Starting with a single D7-brane whose world-volume is extended along $\mathcal{W}_8$, the Wess–Zumino term in the effective action of the D7-brane induces a 3d CS coupling by

$$S_{CS} = \frac{\mu_7 \pi^2 {\ell_{\text{s}}}^4}{24} \int_{\mathcal{W}_8} C^{(4)} \wedge \mathrm{Tr}(\mathcal{R} \wedge \mathcal{R}) \ , \tag{4.64}$$

with $\mu_7 = ((2\pi)^7 {\ell_s}^8 g_s)^{-1}$. The results of [74] allow one to extract the subleading contribution from the coefficient of the Chern-Simons term

$$S_{CS}(\mathrm{AdS}_3) = \frac{c_L - c_R}{96\pi} \int_{\mathrm{AdS}_3} \omega_{CS}(\mathrm{AdS}_3) \ . \tag{4.65}$$

One should then sum over all the 7-branes in the solution.

The number (and type of) 7-branes in the background are encoded in the elliptic fibration. In the simplest case of an elliptic surface $\mathbb{E}_\tau \hookrightarrow \mathcal{S}^\tau \to \Sigma$ the number of 7-branes, assuming only I_1 fibers, is given by $12\deg(\mathcal{L}_D)$. The canonical bundle of the total space of an elliptic surface is

$$K_{\mathcal{S}^\tau_4} = \pi^* \left(K_\Sigma + \sum_{i=1}^{|\Delta|} a_i P_i \right) \ , \tag{4.66}$$

where i is summed over the components of the discriminant Δ of the elliptic fibration and a_i are coefficients determined by the type of the singular fibers and π is the projection to the base. For I_1 fibers as considered here $a_i = \frac{1}{12}$. In order to satisfy

$$\mathfrak{R}_{\mathcal{S}^\tau_4} = -3J_\Sigma = -K_{\mathcal{S}^\tau_4} \ , \quad \text{and} \quad \mathfrak{R}_\Sigma = -3J_\Sigma - \mathrm{d}Q = -K_\Sigma \, , \tag{4.67}$$

one obtains that the number of I_1 fibers is

$$|\Delta| = 12\deg(\mathcal{L}_D) \, . \tag{4.68}$$

Notice that for an elliptically fibered K3 surface, whose base is necessarily a $\mathbb{P}^1$, $\deg(\mathcal{L}_D) = c_1(\mathbb{P}^1) = 2$ implies the well-known result of 24 7-branes.

We will also compare R-charges and baryonic charges in the holographic duality. Recall that in the Sasaki–Einstein setup one may compute these by evaluating the volumes of certain supersymmetric three cycles $\{\Sigma_i\}$. Below we present a version of this computation in the context of the AdS_3 solutions of interest. We assume that, similarly to their AdS_5 counterparts, D3-branes wrapped on Σ_i give rise to BPS particles moving in AdS_3, which we conjecture to be dual to some baryonic-type operator in the CFT_2. These are spin-0 BPS objects, and in 2d their conformal dimension equals their R-charge. Denoting by $\mathcal{B}_{\Sigma_i}$ the operators in the dual field theory associated to the three-cycle Σ_i, the conformal dimension is

$$R[\mathcal{B}_{\Sigma_i}] = \Delta[\mathcal{B}_{\Sigma_i}] = \frac{M[\mathcal{B}_{\Sigma_i}]}{m} \ , \tag{4.69}$$

where $M[\mathcal{B}_{\Sigma_i}]$ is the mass of the wrapped D3-brane. As our solutions include a warp factor for AdS_3 depending on the internal manifold, the mass of the D3-brane wrapped on the three-cycle Σ_i, is given by

$$M[\mathcal{B}_{\Sigma_i}] = T_3 \int_{\Sigma_i} \frac{\mathrm{e}^H}{m} \mathrm{dvol}(\Sigma_i) \ , \qquad T_3 = \frac{1}{8\pi^3 {\ell_\mathrm{s}}^4 g_\mathrm{s}} \ , \tag{4.70}$$

where T_3 is the D3-brane tension. The factor of $\frac{\mathrm{e}^H}{m}$ is precisely the warp factor due to the warping of the time coordinate. In summary

$$R[\mathcal{B}_{\Sigma_i}] = \frac{2\pi N}{m^4} \frac{\int_{\Sigma_i} \mathrm{e}^{4H} \widehat{\mathrm{dvol}}(\Sigma_i)}{\int_{\mathcal{M}_5} F} \ . \tag{4.71}$$

The volume form with a hat is defined to be the volume form of the unwarped dimensionless metric obtained from the bracketed expression in (2.51). Notice the similarity with the formulas for geometric R-charge in warped AdS_4 backgrounds [101, 102].

The supersymmetric cycles are divisors in the complex cone over $\mathcal{M}_7$, which implies that they are calibrated with respect to the four-form $\frac{\mathrm{e}^{4H}}{2r^2} J_\mathrm{cone} \wedge J_\mathrm{cone}$, with J_cone the Kähler form on the 8d metric cone $\mathrm{d}s^2_\mathrm{cone} = \mathrm{d}r^2 + r^2 \mathrm{d}s^2(\mathcal{M}_7)$. Recall that unlike in the Sasaki–Einstein case, the cone is neither Ricci-flat nor Kähler, but instead, as follows from [103], satisfies

$$\mathrm{d}(r^{-4} \mathrm{e}^{8H} J_\mathrm{cone} \wedge J_\mathrm{cone} \wedge J_\mathrm{cone}) = 0 \ . \tag{4.72}$$

In fact for all the solutions presented above a stronger condition holds. In each of the solutions presented above there is a distinguished Riemann surface. Define $\tilde{J}$ to be the Kähler form at fixed coordinate on the cone, then we have

$$\mathrm{d}(r^{-2} \mathrm{e}^{4H} \tilde{J} \wedge \tilde{J}) = 0 \ . \tag{4.73}$$

It follows that the three-cycles are calibrated with respect to the above form and therefore they are supersymmetric cycles.

The final holographic charges that we can compute are the baryonic charges. In particular, we shall use the observation that the integral of a harmonic three-form over each of the three-cycles gives the baryonic charges of each of the baryons dual to that cycle in the field theory up to some overall normalisation which is fixed by requiring the results are integer. We note that as this result is a topological invariant we are free to multiply the metric by an arbitrary bounded and non-vanishing warp factor and perform the computation using the warped metric. We shall make use of this freedom later.

4.2.2 Universal Twist Solutions: Elliptic Surface Case

Consider first the universal twist solutions, where $\mathcal{Y}_8^\tau$ has an elliptic surface factor

$$\mathrm{AdS}_3 \times S^1 \to (\mathcal{M}_4 \times \mathcal{S}_4^\tau)\,, \qquad \mathbb{E}_\tau \hookrightarrow \mathcal{S}_4^\tau \to \Sigma\,. \tag{4.74}$$

Recall also that for a fixed coordinate on Σ the transverse space is a Sasaki–Einstein manifold $SE_5 = (S^1 \to \mathcal{M}_4)$.

Central Charges

We first consider the holographic charges of the universal twist solution with τ varying over Σ. From (4.63) we have

$$\begin{aligned} c_{\text{sugra}} &= \frac{2\cdot 3^4\pi^2}{((2\pi m\ell_{\text{s}})^4 g_{\text{s}})^2}\mathrm{vol}(\mathrm{SE}_5)\mathrm{vol}(\Sigma) = \frac{2\pi^2 N^2 \mathrm{vol}(\Sigma)}{\mathrm{vol}(SE_5)} \\ &= \frac{36 n^2 M}{\tilde{m}^2 h^2 \ell}(2(g-1) + \deg(\mathcal{L}_{\mathcal{D}}))\,, \end{aligned} \tag{4.75}$$

where we have used the quantisation conditions in (4.26). In the final step we have re-expressed the volume of Σ, by using the fact that the Ricci form on Σ satisfies (4.17), as

$$\mathrm{vol}(\Sigma) = \int_\Sigma J_\Sigma = -\frac{1}{3}\left(\int_\Sigma \mathfrak{R}_\Sigma + \mathrm{d}Q\right) = -\frac{1}{3}\left(4\pi(1-g) + \int_\Sigma \mathrm{d}Q\right)\,, \tag{4.76}$$

and using

$$-\frac{1}{2\pi}\int_\Sigma \mathrm{d}Q = \int_\Sigma c_1(\mathcal{L}_D) = \deg(\mathcal{L}_D)\,, \tag{4.77}$$

we have

$$\mathrm{vol}(\Sigma) = \frac{1}{3}\left(4\pi(g-1) + 2\pi\deg(\mathcal{L}_D)\right)\,. \tag{4.78}$$

Moreover it follows that the central charge is integer for any Kähler–Einstein base and any surface Σ. To make contact with the field theory this can be related to the "a" central charge of the 4d quiver theory dual to the Sasaki–Einstein solution (with constant τ) as

$$c_{\text{sugra}} = \frac{8a^{4d}}{\pi}\mathrm{vol}(\Sigma)\,, \qquad \text{where} \quad a^{4d} = \frac{N^2\pi^3}{4\mathrm{vol}(SE_5)}\,. \tag{4.79}$$

we conclude that at leading order in N the central charge is integral and given by

$$c_{\text{sugra}} = a^{4d}\left[\frac{32(g-1)}{3} + \frac{16}{3}\deg(\mathcal{L}_D)\right]\,. \tag{4.80}$$

The first term is precisely the result one obtains for the constant τ solution. Notice that even at leading order there is a correction to the central charge due to the varying axio-dilaton τ, proportional to the first Chern class of the $U(1)_D$ duality bundle.

We note that this central charge is integer, independent of the choice of Kähler-Einstein base and curve Σ. To see this one should consider the last expression in (4.75). There are three possible choices for Kähler-Einstein base; $\mathbb{CP}^2$ with $(M, \tilde{m}) = (9, 3)$, $S^2 \times S^2$ with $(M, \tilde{m}) = (8, 2)$ and $\mathrm{d}P_k$ for $k = 3, .., 8$ with $(M, \tilde{m}) = (9-k, 1)$. Simple numerology shows that (4.75) is integer for any of these choices and therefore also (4.80).

By using (4.64) and (4.65) we find the contribution of a single 7-brane to the difference of central charges is

$$\Delta((c_L)_{\text{sugra}} - (c_R)_{\text{sugra}}) = \frac{N}{2} \ . \tag{4.81}$$

Therefore the total contribution from the 7-branes is given by

$$(c_L)_{\text{sugra}} - (c_R)_{\text{sugra}} = (\text{number of 7-branes}) \cdot \frac{N}{2} = 6N \deg(\mathcal{L}_D) \ . \tag{4.82}$$

R-charges

Recall that at fixed coordinates on Σ the $U(1)$-fibration over the Kähler–Einstein space $\mathcal{M}_4$ is a Sasaki–Einstein manifold, therefore the three-cycles which are dual to baryonic operators in 2d are the same as those in 4d[6]. From (4.71) the R-charges are

$$R[\mathcal{B}_{\Sigma_i}] = 2\pi N \frac{\int_{\Sigma_i} \mathrm{e}^{4H} \left(\frac{9}{4}\right)^{\frac{3}{2}} \mathrm{d}\widetilde{\mathrm{vol}}(\Sigma_i)}{9\widetilde{\mathrm{vol}}(SE_5)} = \frac{N\pi}{3} \frac{\int_{\Sigma_i} \mathrm{d}\widetilde{\mathrm{vol}}(\Sigma_i)}{\widetilde{\mathrm{vol}}(SE_5)} = R^{4d}[\mathcal{B}_{\Sigma_i}] \ , \tag{4.83}$$

where we have used [33, 104] to compare with the corresponding 4d R-charge.

Baryonic Charges

During the discussion on baryonic charges we noted that the result is independent of a rescaling of the metric. Clearly this implies that the baryonic charges for these solutions will be identical to the original AdS_5 computation and therefore we shall not present it.

[6] The Sasaki–Einstein metric (at fixed Σ coordinates) appearing in the AdS_3 solution has a constant rescaling in comparison with the AdS_5 metric and therefore the volume form on the any three-cycle differs by a factor of $\mathrm{e}^{3H}\left(\frac{9}{4}\right)^{3/2}$ in comparison with the AdS_5 normalised metric. We shall write all volume forms with respect to the canonically normalised metric on the Sasaki–Einstein manifold with a tilde.

Holographic charge	Result
c_{sugra}	$\frac{32(g-1)a^{4d}}{3} + \frac{16a^{4d}}{3}\deg(\mathcal{L}_D)$
$(c_L)_{\text{sugra}} - (c_R)_{\text{sugra}}$	$6N\deg(\mathcal{L}_D)$
R-charges	$R^{(2d)}[\mathcal{B}_{\Sigma_i}] = R^{(4d)}[\mathcal{B}_{\Sigma_i}]$
Baryonic charges	$B^{(2d)}[\mathcal{B}_{\Sigma_i}] = B^{(4d)}[\mathcal{B}_{\Sigma_i}]$

Table 4.1: Holographic charges for the universal twist solution with elliptic surface $\mathcal{S}^\tau$. Here, a^{4d} is the 4d central charge (4.79) associated to the dual of the $\mathrm{AdS}_5 \times SE_5$ solutions.

4.2.3 Universal Twist Solutions: Elliptic Three-fold Case

Consider now the universal twist solution where $\mathcal{Y}_8^\tau$ has a factor given by an elliptic three-fold. It will be instructive to compare these solutions to the AdS_5 solutions in section 2.4.3 in an analogous manner to the way in which the discussion in the previous section referenced the Sasaki–Einstein solutions.

Central Charges

The leading order central charge is easily found to be

$$c = \frac{2 \cdot 3^4 \pi^2}{((2\pi m \ell_s)^4 g_s)^2} \mathrm{vol}(\mathcal{M}_5^\tau)\mathrm{vol}(\Sigma) = \frac{8\pi^3(g-1)N^2}{3\mathrm{vol}(\mathcal{M}_5^\tau)} = \frac{32(g-1)}{3} a_\tau^{4d} \ , \tag{4.84}$$

where a_τ^{4d} is the central charge of the τ dependent 4d field theory dual to the solutions discussed in section 2.4.3.

As in the previous cases, the subleading contribution to the difference of central charges can be determined by anomaly inflow on the 7-branes, from the Wess–Zumino term (4.64) in the effective action of a single D7-brane. In contrast to the first case, the discriminant locus of the elliptic fibration is now a curve in $\mathcal{M}_4$. We consider only I_1 singular fibers and thus only single 7-branes are wrapped on curves C_x in the discriminant locus[7]. Δ. Imposing that the elliptic fibration satisfies (4.29) implies

$$[\Delta] = \sum_x \omega_x = 12 c_1(\mathcal{L}_D) \ , \tag{4.85}$$

where ω_x are the two-forms dual to the curves C_x, which are wrapped by the single 7-branes. Each 7-brane is extended along $\mathrm{AdS}_3 \times (U(1)_\chi \to \Sigma \times C_x)$, where χ is the angular coordinate with period $2\pi\ell$ along the R-symmetry direction. The total contribution to the WZ term is obtained by summing over all the single 7-branes, so that the effective world-volume can be written as $\mathcal{W}_8 = \mathrm{AdS}_3 \times (\mathcal{M}_3 \to \Sigma)$, where $\mathcal{M}_3 = U(1)_\chi \to \Delta$ is a three-cycle in $\mathcal{M}_5^\tau$. The three-dimensional Chern-Simons

[7]With a slight abuse of notation we denote simply as Δ the locus $\{\Delta = 0\}$.

term arising from the Wess-Zumino action then reads[8]

$$
\begin{aligned}
S_{CS} &= -\tilde{\mu}_7 \int_{\mathcal{W}_8} F \wedge \omega_{CS}(\mathrm{AdS}_3) \\
&= -\frac{3\tilde{\mu}_7}{4m^4}\, 2\pi\chi(\Sigma)\, \mathrm{vol}(\mathcal{M}_3) \int_{\mathrm{AdS}_3} \omega_{CS}(\mathrm{AdS}_3)\ ,
\end{aligned} \tag{4.86}
$$

where

$$
\begin{aligned}
\mathrm{vol}(\mathcal{M}_3) &= \int_{\mathcal{M}_3} \frac{1}{3}(\mathrm{d}\chi + \rho) \wedge J_{\mathcal{M}_4} = \frac{2\pi\ell}{3} \int_{\Delta} J_{\mathcal{M}_4} \\
&= 8\pi\ell \int_{\mathcal{M}_4} J_{\mathcal{M}_4} \wedge c_1(\mathcal{L}_D) = \frac{8\pi^2\ell}{3} \int_{\mathcal{M}_4} \left(c_1(\mathcal{M}_4) \wedge c_1(\mathcal{L}_D) - c_1(\mathcal{L}_D)^2 \right)\ .
\end{aligned} \tag{4.87}
$$

The gravitational anomaly, by using (4.65), is therefore found to be

$$
c_L - c_R = -\frac{2^4 3^2 \pi^2 (g-1)\tilde{\mu}_7 \mathrm{vol}(\mathcal{M}_3)}{m^4}\ , \tag{4.88}
$$

where notice that $\mathrm{vol}(\mathcal{M}_3)$ is essentially an intersection number, providing the effective number of 7-branes, as in [59].

We will relate $c_L - c_R$ in the dual 2d SCFT to a corresponding holographic quantity in the parent 4d SCFT, therefore $\mathrm{vol}(\mathcal{M}_3)$ will drop out from the equation. Later we will show that this relationship is reproduced exactly by a field theoretic calculation, although we will not attempt to calculate the precise values of the 4d central charges in specific examples.

Concretely, we wish to identify the above result with the linear 't Hooft anomaly k_R in the 4d theory, and therefore with the difference of 4d central charges $c^{4d} - a^{4d} = \frac{k_R}{16}$, where recall that

$$
a^{4d} = \frac{3}{32}(3k_{RRR} - k_R)\ , \qquad c^{4d} = \frac{1}{32}(9k_{RRR} - 5k_R)\ . \tag{4.89}
$$

For any 4d $\mathcal{N} = 1$ SCFT with an R-symmetry, the R-symmetry current $\mathcal{R}_\mu$ satisfies the anomalous conservation equation [21, 105, 106]

$$
\partial_\mu \langle \sqrt{g} \mathcal{R}^\mu \rangle = \frac{k_R}{384\pi^2} \epsilon_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\kappa\tau} R^{\rho\sigma\kappa\tau} + \frac{k_{RRR}}{48\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\ , \tag{4.90}
$$

where F is the field strength of the background gauge field A sourcing the R-symmetry current.

Consider the AdS_5 solutions of section 2.4.3. Recall that for the universal twist solution to be well-defined the manifold $\mathcal{M}_5^\tau$ is required to be quasi-regular. As such

[8] In the following discussion the overall constant of the Wess-Zumino term in equation (4.64) will cancel in the computation and therefore for simplicity we define the new constant $\tilde{\mu}_7 = \frac{\mu_7 \pi^2 \ell_s{}^4}{24}$.

we may write the metric on $\mathcal{M}_5^\tau$ as a $U(1)$ fibration over a Kähler base $\mathcal{M}_4$ as

$$\mathrm{d}s^2(\mathcal{M}_5^\tau) = \frac{1}{9}\left(\mathrm{d}\chi + 3\sigma\right)^2 + \mathrm{d}s^2(\mathcal{M}_4) \ , \tag{4.91}$$

with $\mathrm{d}\sigma = 2J_{\mathcal{M}_4}$. As we consider only the quasi-regular cases we may fix the period of χ to be $2\pi\ell$. By changing coordinates as $\chi = \ell\tilde{\chi}$ we define a new 2π periodic coordinate. As the Reeb vector field is dual to the R-symmetry direction it is natural, as explained in [104], that a shift in the coordinate $\tilde{\chi}$ induces a gauge transformation of the R-symmetry gauge field[9] $\mathcal{A}$, that is

$$\tilde{\chi} \to \tilde{\chi} + \alpha\Lambda \ , \qquad \mathcal{A} \to \mathcal{A} + \mathrm{d}\Lambda \ . \tag{4.92}$$

The identification of the constant α is fixed by using the fact that the holomorphic 3-form on the cone is associated to the superpotential and therefore has R-charge 2. The functional dependence of the holomorphic three-form on $\tilde{\chi}$ may be read off from

$$\partial_{\tilde{\chi}}\Omega = 3\mathrm{i}\Omega \ , \tag{4.93}$$

which fixes $\alpha = \frac{2}{\ell}$. We may include $\mathcal{A}$ in the usual Kaluza-Klein ansatz by deforming the internal metric as

$$\mathrm{d}s^2(\mathcal{M}_5^\tau) \to \left(\frac{\ell}{3}\right)^2 \left(\mathrm{d}\tilde{\chi} + \frac{3}{\ell}\sigma + \frac{2}{\ell}\mathcal{A}\right)^2 + \mathrm{d}s^2(\mathcal{M}_4) \ . \tag{4.94}$$

Moreover, for consistency, the five-form flux must be deformed as[10]

$$F \to (1+*)\frac{2\ell}{3m_5^4}\left(\left(\mathrm{d}\tilde{\chi} + \frac{3}{\ell}\sigma + \frac{2}{\ell}\mathcal{A}\right)\wedge J_{\mathcal{M}_4}\wedge J_{\mathcal{M}_4} - \frac{1}{3}\mathrm{d}\mathcal{A}\wedge\left(\mathrm{d}\tilde{\chi} + \frac{3}{\ell}\sigma\right)\wedge J_{\mathcal{M}_4}\right) \ , \tag{4.95}$$

which by construction is closed upon using the equation of motion for the new gauge field $\mathrm{d}*\mathrm{d}\mathcal{A} = 0$. The term of the four-form potential of interest is

$$C_4 \supset -\frac{2}{3m_5^4}\mathcal{A}\wedge\left(\frac{\ell}{3}\left(\mathrm{d}\tilde{\chi} + \frac{3}{\ell}\sigma\right)\right)\wedge J_{\mathcal{M}_4} \, . \tag{4.96}$$

In this configuration, the world-volume of each 7-brane is $\mathrm{AdS}_5 \times (U(1)_\chi \to C_x)$, therefore the total contribution from all the 7-branes is obtained by integrating on the world-volume $\mathcal{W}_8 = \mathrm{AdS}_5 \times \mathcal{M}_3$ where $\mathcal{M}_3 = U(1)_\chi \to \Delta$ is the same three-cycle in $\mathcal{M}_5^\tau$ that appears in the AdS_3 solution. We may use this to extract from

[9]More precisely, here $\mathcal{A}$ is a gauge field in AdS_5, whose boundary value is identified with the background R-symmetry gauge field A in the four dimensional SCFT.

[10]The length scale associated to the AdS_5 will be denoted as m_5 in the following. It will be shown to be proportional to the length scale m in the AdS_3 solutions.

the Wess-Zumino term a contribution to the gravitational action in AdS_5 given by

$$S_{CS} = \tilde{\mu}_7 \int_{\mathcal{W}_8} C_4 \wedge \mathrm{Tr}[\mathcal{R} \wedge \mathcal{R}] = -\frac{2\tilde{\mu}_7}{3m_5^4}\mathrm{vol}(\mathcal{M}_3) \int_{\mathrm{AdS}_5} \mathcal{A} \wedge \mathrm{Tr}[\mathcal{R} \wedge \mathcal{R}] \ . \qquad (4.97)$$

According to the gauge/gravity duality master formula, the generating functional for (connected) current correlators in the boundary theory, $\mathrm{i}W[A] = \log Z[A]$, equates the on-shell gravitational action, $W[A] = S_{\mathrm{AdS}_5}[\mathcal{A}]$, and therefore as explained in [24] the non-invariance under gauge transformations of the latter corresponds to the anomaly in the dual field theory. Specifically, a gauge transformation of the boundary gauge field A induces a transformation of the Chern-Simons term

$$\begin{aligned} \delta_\Lambda W[A] = \delta_\Lambda S_{CS} &= -\frac{2\tilde{\mu}_7}{3m_5^4}\mathrm{vol}(\mathcal{M}_3) \int_{\mathrm{AdS}_5} \mathrm{d}\Lambda \wedge \mathrm{Tr}[\mathcal{R} \wedge \mathcal{R}] \\ &= -\frac{2\tilde{\mu}_7}{3m_5^4}\mathrm{vol}(\mathcal{M}_3) \int_{\partial\mathrm{AdS}_5} \Lambda \mathrm{Tr}[\mathcal{R} \wedge \mathcal{R}] \ , \end{aligned} \qquad (4.98)$$

implying that on the boundary we have

$$\int_{\partial\mathrm{AdS}_5} \Lambda \partial_\mu \langle \sqrt{g} \mathcal{R}^\mu \rangle \mathrm{dvol}(\partial\mathrm{AdS}_5) = \frac{2\tilde{\mu}_7}{3m_5^4}\mathrm{vol}(\mathcal{M}_3) \int_{\partial\mathrm{AdS}_5} \Lambda \mathrm{Tr}[\mathcal{R} \wedge \mathcal{R}] \, , \qquad (4.99)$$

where

$$\mathrm{Tr}[\mathcal{R} \wedge \mathcal{R}] = -\frac{1}{4}\epsilon_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\kappa\tau} R^{\rho\sigma\kappa\tau} \mathrm{dvol}(\partial\mathrm{AdS}_5) \ . \qquad (4.100)$$

In conclusion we find[11]

$$k_R = -N \frac{\pi \mathrm{vol}(\mathcal{M}_3)}{12 \mathrm{vol}(\mathcal{M}_5^\tau)} = -\frac{2^6 \pi^2 \tilde{\mu}_7 \mathrm{vol}(\mathcal{M}_3)}{m_5^4} \ , \qquad (4.101)$$

and inserting this into (4.88) we obtain

$$c_L - c_R = \frac{9}{4}\frac{m_5^4}{m^4}(g-1)k_R \ . \qquad (4.102)$$

We may relate the different length scales of the two solutions by comparing the quantisation condition used to obtain the integer N. In both cases this gives the number of D3-branes in the solution and should therefore be fixed in flowing from the AdS_5 solution to the AdS_3 solution, by comparing (A.98) and (4.32) we find $9m_5^4 = 4m^4$ and therefore we conclude that

$$c_L - c_R = (g-1)k_R \ . \qquad (4.103)$$

[11] It would be interesting to match this formula, using (4.33) and (4.87), to a purely field theoretic computation in the 4d SCFT.

R-charges

In a similar manner to the previous section, at fixed coordinate on Σ, which is now $\mathbb{H}^2/\Gamma$ with Γ a subgroup of $SL(2,\mathbb{Z})$, equipped with the constant curvature metric, one finds that the metric on $\mathcal{M}_5^\tau$ is the same (up to an overall constant factor) as the metrics discussed in section 2.4.3. Again we have that the three-cycles of the two solutions agree and therefore the dual baryonic operators in 2d and 4d are identified. Clearly by the same arguments as presented in section 4.2.2 the R-charges of the baryonic operators in 2d and 4d coincide.

Baryonic Charges

As above the metrics agree up to a numerical factor. The topological nature of this computation implies that the baryonic charges of the 2d theory and the 4d theory agree.

Holographic charge	Result
c_{sugra}	$\frac{32(g-1)}{3}a_\tau^{4d}$
$(c_L)_{\text{sugra}} - (c_R)_{\text{sugra}}$	$16(g-1)(c^{4d}-a^{4d})$
R-charges	$R^{(2d)}[\mathcal{B}_{\Sigma_i}] = R^{(4d)}[\mathcal{B}_{\Sigma_i}]$
Baryonic Charges	$B^{(2d)}[\mathcal{B}_{\Sigma_i}] = B^{(4d)}[\mathcal{B}_{\Sigma_i}]$

Table 4.2: Holographic charges for the universal twist with elliptic threefold $\mathcal{T}_6^\tau$. Here a_τ^{4d} is the central charge of the dual to the solutions in section 2.4.3.

4.2.4 Baryonic Twist Solution: $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ Case

In this final section we shall consider the baryonic twist solutions using $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ as the example. We expect the computations to extend to other solutions with baryonic twists in a similar manner.

Central Charges

From (4.63) we compute

$$c_{\text{sugra}} = \frac{6NM\mathfrak{p}^2(\mathfrak{q}^2-\mathfrak{p}^2)}{\mathfrak{q}^2}\ . \tag{4.104}$$

Notice that despite the differences of our solution with respect to the constant τ version discussed in [56], the value of the holographic central charge (4.104) agrees *exactly* with the value obtained in eq. (4.18) of [56]. We anticipate that this is a general property of the baryonic twist solutions, that does not depend on the details of the $\mathcal{M}_5^\tau$ geometry. More precisely, for any solution of the type $\text{AdS}_3 \times T^2 \times \mathcal{M}_5^\tau$

and constant axio-dilaton, we can construct a solution of the type $\mathrm{AdS}_3 \times \mathbb{P}^1 \times \mathcal{M}_5^\tau$ with axio-dilaton varying holomorphically on $\mathbb{P}^1$, such that the F-theory lift has an elliptic K3 factor. These two solutions will have equal holographic central charges, at leading order in N.

The subleading contribution may also be simply computed from the geometry. Moreover it can be seen that the result is independent of the choice of $\mathcal{M}_5^\tau$, one obtains the universal contribution of $\frac{N}{2}$ for a single 7-brane. For a K3 surface the number of 7-branes for a consistent geometry is 24 and therefore the subleading contribution is

$$(c_L)_{\text{sugra}} - (c_R)_{\text{sugra}} \;=\; 24 \cdot \frac{N}{2} \;=\; 12N \; . \tag{4.105}$$

Notice that although at leading order the central charges of the solution with constant and varying τ agree, the subleading contribution (4.105) is clearly zero in the model with constant τ, as there are no seven-branes. In the next section we will argue that in the dual field theory side this result is exactly reproduced combining contributions that come both from the bulk modes (3-3 strings) as well as 3-7 strings. Note that there are $O(N)$ terms in the bulk for the theory with varying coupling, that are due to the duality twisting.

R-charges

The three-cycles in the geometry that are calibrations are the four three-cycles located at the four degeneration surfaces of the metric. Recall that at each degeneration surface a Killing vector has zero norm, which determines a codimension two locus (namely a three-manifold) in the geometry. By explicit computation one can see that the volume form on the three-manifolds will be equal to the pullback of this closed four-form and hence these cycles are calibrated. We find

$$\begin{aligned} R[\mathcal{B}_{\Sigma_+}] = R[\mathcal{B}_{\Sigma_-}] = N\frac{\mathfrak{q}^2 - \mathfrak{p}^2}{\mathfrak{q}^2} \;, \\ R[\mathcal{B}_{\Sigma_0}] = R[\mathcal{B}_{\Sigma_\pi}] = N\frac{\mathfrak{p}^2}{\mathfrak{q}^2} \;. \end{aligned} \tag{4.106}$$

Observe that the sum of the normalised volumes of submanifolds satisfies exactly the same relation to their Sasaki–Einstein counterparts, namely

$$R[\mathcal{B}_{\Sigma_+}] + R[\mathcal{B}_{\Sigma_-}] + R[\mathcal{B}_{\Sigma_0}] + R[\mathcal{B}_{\Sigma_\pi}] = 2N \; . \tag{4.107}$$

It would be nice to obtain a general proof of this formula, analogous to that in [30].

Baryonic Charges

As discussed in the introduction of this section we are free to multiply the metric by an arbitrary warp factor so long as the warp factor is bounded and non-vanishing. We shall make use of this freedom to find such a harmonic form. As $\dim[H^3(\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}})] = 1$ there is a unique closed three-form representative which may be extracted from (4.52) and is given by

$$\omega_3 = k\left(D\alpha \wedge D\phi \wedge \mathrm{d}x + \mathrm{dvol}(S^2) \wedge \left(xD\alpha - \frac{U(x)}{w(x)}D\phi\right)\right) \ . \tag{4.108}$$

Observe that for the warped metric on $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$

$$\mathrm{d}s^2 = \mathrm{e}^{-4H}\mathrm{d}s^2(\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}) \tag{4.109}$$

this three-form is both closed and co-closed and therefore harmonic. The constant k, fixed by requiring that the results are integer, is $k = -\frac{\mathfrak{q}^2-\mathfrak{p}^2}{4}$. Integrating this over the three-cycles we find

$$\int_{\Sigma_\pi}\omega_3 = \int_{\Sigma_0}\omega_3 = \mathfrak{p} \ , \quad \int_{\Sigma_-}\omega_3 = \mathfrak{q}-\mathfrak{p} \ , \quad \int_{\Sigma_+}\omega_3 = -(\mathfrak{q}+\mathfrak{p}) \tag{4.110}$$

which gives the baryonic charges of the fields and agrees with the result in (C.41) for the would-be GLSM charges.

Holographic charge	Result
c_{sugra}	$\frac{6NM\mathfrak{p}^2(\mathfrak{q}^2-\mathfrak{p}^2)}{\mathfrak{q}^2}$
$(c_L)_{\text{sugra}} - (c_R)_{\text{sugra}}$	$12N$
R-charges	$R^{(2d)}[\mathcal{B}_{\Sigma_0}] = R^{(2d)}[\mathcal{B}_{\Sigma_\pi}] = N\frac{\mathfrak{q}^2-\mathfrak{p}^2}{\mathfrak{q}^2}$ $R^{(2d)}[\mathcal{B}_{\Sigma_0}] = R^{(2d)}[\mathcal{B}_{\Sigma_\pi}] = N\frac{\mathfrak{p}^2}{\mathfrak{q}^2}$
Baryonic charges	$B^{(2d)}[\mathcal{B}_{\Sigma_0}] = B^{(4d)}[\mathcal{B}_{\Sigma_\pi}] = \mathfrak{p}$ $B^{(2d)}[\mathcal{B}_{\Sigma_0}] = \mathfrak{q}-\mathfrak{p}$ $B^{(2d)}[\mathcal{B}_{\Sigma_0}] = -(\mathfrak{q}+\mathfrak{p})$

Table 4.3: Holographic charges of the Baryonic twist for $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$.

4.3 Dual Field Theories with Varying Coupling

Field theories with spacetime varying coupling are not a new concept in itself. However, the inclusion of S-duality monodromies specifically in 4d $\mathcal{N}=4$ SYM and, as we will see, generalizations to $\mathcal{N}=1$, have only received attention recently in [16,18].

We saw previously that the field theory central charge in the $(0,4)$ case was determined by the UV spectrum. For $(0,2)$ SCFTs in 2d the $U(1)_R$ R-symmetry mixes with global symmetries along the RG flow and one must invoke c-extremization to compute the central charges [22] (see also [107]) in the IR.

In the remainder of this section we shall discuss in general field theories with varying couplings and c-extremization before using these techniques to match the holographic charges obtained previously.

4.3.1 Duality Twist for 4d $\mathcal{N} = 4$ SYM

For 4d $\mathcal{N} = 4$ the question arose in the context of D3-branes in F-theory, which naturally implements the varying complexified coupling τ in terms of a complex structure of an elliptic curve. Field theoretically the τ variation along a curved manifold, e.g. a complex curve or surface, together with retaining some supersymmetry, implies that a particular new topological twist needs to be applied to the field theory. This topological duality twist was first introduced for abelian theories in [15], and a proposal for the non-abelian generalization was put forward based on a realization in terms of M5-branes in [16]. For D3-branes wrapped on curves along which the coupling varies, the duality twist was implemented in [18, 19].

The key point about the topological duality twist is that fields and supercharges transform as sections of a duality bundle $\mathcal{L}_D$ with connection given in terms of $\tau = \tau_1 + i\tau_2$ by Q in (1.8). The transformation of the supercharges is such that they have charge $\pm 1/2$ under this $U(1)_D$:

$$\begin{aligned} Q_\alpha &\rightarrow \mathrm{e}^{-i\alpha(\gamma)} Q_\alpha \\ \tilde{Q}_{\dot\alpha} &\rightarrow \mathrm{e}^{+i\alpha(\gamma)} \tilde{Q}_{\dot\alpha} \end{aligned} \tag{4.111}$$

where $e^{i\alpha(\gamma)} = (c\tau + d)/|c\tau + d|$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$. The remaining fields of the $\mathcal{N} = 4$ SYM theory are charged $q_\Phi = 0$ (scalars), $q_{F_\pm} = \mp 1$ (where $F_\pm = \sqrt{\tau_2}(F \pm \star F)/2$) and $q_\lambda = -\frac{1}{2}$, $q_{\tilde\lambda} = \frac{1}{2}$ (fermions). To offset this transformation the duality twist redefines the $U(1)_D$ with an R-symmetry transformation. More generally for spacetimes of the form $\mathcal{M}_4 = \mathbb{R}^{1,1} \times C$ the twist can involve $U(1)_C, U(1)_D$ and an R-symmetry $U(1)_R$, as discussed in [18].

One of the classes of solutions that we will encounter is the compactification of a 4d $\mathcal{N} = 1$ theory on a curve $C = \mathbb{P}^1$, which is the base of an elliptic K3. This has many similarities to the elliptic K3 compactifications of F-theory as discussed in appendix D of [18]. Briefly, in this case the twist only requires one to combine

$$U(1)_{\text{twist}} : \qquad T^C_{\text{twist}} = \frac{1}{2}\left(T_C - T_D\right) , \tag{4.112}$$

without an R-symmetry twist. The resulting theory has 2d $(0, 8)$ supersymmetry. The fields are counted by cohomologies $h^{i,j}(C)$, depending on the twist charges $q_{\text{twist}} = -1, 0, +1$ corresponding to $(i, j) = (1, 0), (0, 0), (0, 1)$.

The analysis for strings arising from wrapped branes was largely performed for abelian theories. The generalisation to non-abelian is somewhat more subtle, and needs to be performed using the approach in [16], mapping the issue to M5-branes on an elliptic surface $\widehat{C}$, which geometrizes the axio-dilaton variation in terms of the elliptic fiber.These theories have been further studied in [108] using anomaly arguments.

For $\mathcal{N} = 1$ theories in 4d, similar compactifications with spacetime dependent couplings can be defined. Although not every such theory has a duality group, whenever there is a holographic dual setup, and an embedding into Type IIB (or F-theory), the theory should have an induced $U(1)_D$ symmetry. One way to argue for this is presented in [34] by Intriligator, where the so-called bonus-$U(1)$, which for the abelian theories was identified with $U(1)_D$ in [16]. Again there is a question of how to generalise this to non-abelian theories, where there is no manifest way to define this duality symmetry. We should remark that this symmetry for the abelian theory is a symmetry only of the equations of motion, not of the action. From considerations in [34], the bonus symmetry is an approximate symmetry only for certain observables in a particular limit, namely when both stringy and D-stringy corrections are suppressed, but then should also be a feature of 4d $\mathcal{N} = 1$ theories.

Here we will consider well-known quiver gauge theories with Type IIB Sasaki–Einstein duals, for which we will discuss generalisations of the "universal twist" and "baryonic twist" [58]. The first class of theories is characterised by having rational R-charges in four dimensions, and otherwise unspecified global symmetries; examples include $\mathcal{N} = 4$ SYM and the Klebanov-Witten model, but more generally the theories discussed in section 2.4.3, which are the most general F-theory solutions with AdS_5 factors dual to 4d $\mathcal{N} = 1$ theories. The second class of theories is characterised by having a global baryonic global symmetry, and may have rational or irrational R-charges in four dimensions; our main example will be the $Y^{p,q}$ quivers [29]. In all cases, the R-charges of the 2d SCFTs will be rational.

In the gauge theories each node of the quiver has a complex coupling constant τ_i and the diagonal combination

$$\tau = \sum_i \tau_i \tag{4.113}$$

is identified in the dual supergravity solution with the axio-dilaton of Type IIB. Unlike the case of $\mathcal{N} = 4$ there is no direct way to identify the charges, but we will argue that the fermions are all charged in the same way, exactly as in $\mathcal{N} = 4$ SYM. The argument to support this uses the duality with AdS_5: although the bonus $U(1)$ is not an actual symmetry of the theory, it is a symmetry for large N and for short operators. In the holographic dual these correspond to Kaluza-Klein modes on the compact part of the supergravity solution. As the latter have definite charges under $U(1)_D$, the expectation is that the dual states will also have a well-defined charge.

The state of the art of the KK-spectrum on Sasaki–Einstein manifolds was obtained in [109].

We begin with 4d $\mathcal{N}=1$ with supercharges $Q=(\mathbf{2},\mathbf{1})$ and $\tilde{Q}=(\mathbf{1},\mathbf{2})$ under $SO(1,3)_L$ and reduce them along the curve C

$$\begin{array}{rcl} SO(1,3)_L & \rightarrow & SO(1,1)_L \times U(1)_C \\ (\mathbf{2},\mathbf{1}) & \rightarrow & \mathbf{1}_{++} \oplus \mathbf{1}_{--} \\ (\mathbf{1},\mathbf{2}) & \rightarrow & \mathbf{1}_{+-} \oplus \mathbf{1}_{-+} \,. \end{array} \tag{4.114}$$

The duality charges are conjecturally $q_Q=-1$ and $q_{\tilde{Q}}=+1$. Then performing the topological twist as in (4.112) results in two scalar supercharges of negative chirality (i.e. $\mathbf{1}_{--}$ and $\mathbf{1}_{-+}$ in the above equation). For abelian $\mathcal{N}=1$ theories the multiplets are such that the scalars are uncharged under the $U(1)_D$ and the fermions carry all the same charge, which agrees with that of the supercharges. This is much alike the charges in the $\mathcal{N}=4$ SYM case. For the non-abelian theory, we proposed to study the theory in a mesonic or Coulomb branch, where using anomalies we can determine the central charges, this has been verified in [108].

4.3.2 Twisted $\mathcal{N}=1$ Field Theories

Before addressing the dual field theory interpretation of the solutions we discussed in section 4.1 we review some aspects of the dualities proposed in [58] for the solutions with *constant* τ [31, 100]. We will follow the notation of these references, except, when we discuss the baryonic twist of the $Y^{p,q}$ theories where we will be careful in distinguishing the parameters p, q in the field theories from the parameters $\mathfrak{p}, \mathfrak{q}$ in the gravity solution [31,100]. As we have already mentioned, although these parameters can be formally identified, they turn out to be defined in disjoint domains.

A 4d $\mathcal{N}=1$ field theory can be compactified on a Riemann surface C_g of genus g by performing a topological twist that preserves $\mathcal{N}=(0,2)$ supersymmetry in two dimensions. Although the details of these two-dimensional theories may be complicated to work out, if these flow to $(0,2)$ SCFTs then many of their properties can be inferred by employing the method of c-extremization [17]. In particular, this method allows one to determine the 2d central charge c_R of these theories, starting from the 't Hooft anomalies of their "parent" four-dimensional theories. The most reliable method to implement this is to consider the anomaly polynomial I_6 of the $\mathcal{N}=1$ 4d theory, that can be usually computed exactly starting from the fermionic field content of the 4d theory. This is given by

$$I_6 = \frac{1}{6} k_{IJK} c_1(\mathcal{F}_I) \wedge c_1(\mathcal{F}_J) \wedge c_1(\mathcal{F}_K) - \frac{1}{24} k_I c_1(\mathcal{F}_I) \wedge p_1(T_4) \,, \tag{4.115}$$

where the index I runs over all the $U(1)$ global symmetries of the theory. Here

$c_1(\mathcal{F}_I)$ are the first Chern classes of the different $U(1)_I$ bundles and $p_1(T_4)$ is the first Pontryagin class of the manifold the theory is placed upon. The constants k_{IJK} and k_I are the cubic and linear 't Hooft anomalies which can be determined from the charges of the fermions in the theory, namely

$$k_{IJK} = \mathrm{Tr}[U(1)_I U(1)_J U(1)_K] = \sum_i q_I^i q_J^i q_K^i \ , \qquad k_I = \mathrm{Tr}[U(1)_I] = \sum_i q_I^i \ , \tag{4.116}$$

where q_I^i denotes the charge of the i-th fermion under $U(1)_I$. This can be reduced to the anomaly polynomial I_4 of the 2d theory by integrating it over C_g, which in a similar notation, reads

$$I_4 = \frac{1}{2} k_{IJ} c_1(\mathcal{F}_I) \wedge c_1(\mathcal{F}_J) - \frac{k}{24} p_1(T_2) \, . \tag{4.117}$$

In the $(0,2)$ SCFT we can then read off the central charges c_R and the gravitational anomaly as

$$c_R = 3k_{RR} \, , \qquad c_R - c_L = k \, . \tag{4.118}$$

In general, to compute the k_{IJ} and k one requires information on the spectrum of fermions of the 2d theory, but for theories coming from a parent 4d theory with known 't Hooft anomalies, these can be extracted simply from

$$I_4 = \int_{C_g} I_6 \ . \tag{4.119}$$

The 2d superconformal $U(1)_R$ symmetry is determined by extremizing the trial k_{RR}.

The topological twist can be performed by switching on background gauge fields for the various global symmetries of the 4d theory, with quantised fluxes through C_g. Consider a quiver gauge theory[12] for which the global symmetries are

$$(U(1)_F)^{n_F} \times (U(1)_B)^{n_B} \times U(1)_R^{4d} \ , \tag{4.120}$$

where F stands for flavour and B stands for baryonic symmetries, respectively. The superscript on the R-symmetry-factor emphasises that this is the exact superconformal R-symmetry of the interacting 4d SCFT, determined by a-maximization.

In the notation of [58], the topological twist can be generically performed along[13]

$$T_{\text{twist}} \;\; = \;\; \sum_I^{n_F} b_I T_I + \sum_I^{n_B} B_I T_{B_I} + \frac{\kappa}{2} T_R^{4d} \ , \tag{4.121}$$

where T_I, T_{B_I}, T_R^{4d} are the generators of the respective global symmetries and $\kappa =$

[12] Twisted compatifications of various four-dimensional quiver gauge theories were studied in [110] and further examples of dual supergravity solutions will be discussed in [111].

[13] T_{twist} refers to the combination of symmetry generators that are used to twist the local Lorentz symmetry along the curve.

$1, 0, -1$ for genus $g = 0, 1$, or $g > 1$, respectively[14]. Here b_I, B_I are suitably quantised parameters, and the factor $\frac{\kappa}{2}$ is determined by requiring that the Killing spinors on C_g become constants, as usual. Notice that as the Killing spinors are not charged under the other global symmetries, this particular way of preserving supersymmetry does not fix the parameters b_I, B_I.

Note that when $\kappa \neq 0$ the twisting (4.121) makes sense only when $U(1)_R^{4d}$ is a compact symmetry. In particular, for the $Y^{p,q}$ theories this is true iff $z \equiv \sqrt{4p^2 - 3q^2}$ is an integer and the 4d R-charges are rational numbers. This implies that generically the 2d R-charges will be rational numbers. When $\kappa = 0$ (namely for $C_{g=1} = T^2$) there is no twist by the 4d R-symmetry and therefore one can start from 4d field theories with irrational R-charges. In the next section we will explain a variant of this twisting, in which we can again start from a 4d field theory with irrational R-charges, and nevertheless compactify this on a $C_{g=0} = \mathbb{P}^1$.

The R-symmetry $U(1)_R^{2d}$ of the $(0, 2)$ theory can in general mix with all the global symmetries of the 4d theory[15], namely in terms of generators we have

$$T_{\text{trial}}^{2d} \;=\; \sum_I^{n_F} \epsilon_I T_I + \sum_I^{n_B} \epsilon_{B_I} T_{B_I} + T_R^{4d} \;, \tag{4.122}$$

where $\epsilon_I, \epsilon_{B_I}$ are a priori real numbers that will be determined my extremizing the trial 2d central charge as a function of these parameters. This calculation was performed in [58] for various examples, using the index theorem to count the fermionic zero modes in 2d [112]. As discussed above, however, the computation using the reduction of the anomaly polynomial of the 4d theory is more robust, as there is no need to assume that the theory is weakly coupled (which is not a correct assumption for most $\mathcal{N} = 1$ theories with Sasaki–Einstein duals).

Universal Twist

This twist can be applied to any theory provided the 4d R-charges are rational, and consists in taking

$$T_{\text{twist}} \;=\; \frac{\kappa}{2} T_R^{4d} \;, \tag{4.123}$$

where $\kappa = -1$. Assuming a general parameterization as in (4.122) the outcome of the extremization procedure is that $\epsilon_I = \epsilon_{B_I} = 0$, so that the 2d and 4d R-symmetries are identified[16], namely $R^{2d} = R^{4d}$.

[14] In this equation it is assumed that C_g has constant curvature.

[15] A priori, there can be global symmetries that emerge in the 2d theory. In this case c-extremization (like a-maximization) cannot be used effectively to determine the R symmetry in the IR.

[16] This holds if the 4d 't Hooft anomaly coefficients obey $k_{RRF} = k_F = 0$ and $k_{RRB} = k_B = 0$, which is true for all quiver gauge theories with toric Sasaki–Einstein duals [113].

At leading order in N, this yields the universal relation

$$c_R = c_L = \frac{32}{3}(g-1)a^{4d} \ . \tag{4.124}$$

Recalling that (at leading order in N) in 4d theories

$$a^{4d} = \frac{9}{32}\sum_i (R_i^{4d}-1)^3 \ , \tag{4.125}$$

one sees that (4.124) is indeed equivalent to $R^{2d} = R^{4d}$ and

$$c_R = -3\cdot 2(g-1)\sum_i \left(-\tfrac{1}{2}\right)(R^{4d}-1)(R^{4d}-1)^2 \, , \tag{4.126}$$

where $-\frac{1}{2}(R^{4d}-1)$ is the net number of 2d fermion zero modes associated to each 4d fermion.

The results of section 4.2.2 may be used to compare with the constant τ version presented here by setting $\deg(\mathcal{L}_D) = 0$. We see that, as noted in [58], the central charges match exactly. Moreover we see that the holographic computations for constant τ implies that $c_L - c_R = \mathcal{O}(1)$ as follows from the field theory computation. Finally the results for the R-charges as presented in section 4.2.2 are in agreement with the results from the field theory computation.

Baryonic Twist

Let us now consider theories that possess at least one baryonic symmetry with generator T_B, so that we can twist as

$$T_{\text{twist}} \ = \ BT_B + \frac{\kappa}{2}T_R^{4d} \ , \tag{4.127}$$

and in particular the theories can now be compactified on a torus, $C_1 = T^2$, with $\kappa = 0$. This twist is purely baryonic and for concreteness we focus on the $Y^{p,q}$ theories, which have $n_B = 1$ and $n_F = 2$. One finds that extremizing k_{RR} gives

$$\epsilon_1 = 0 \ , \qquad \epsilon_2 = \frac{p+z}{3q} \ , \qquad \epsilon_B = \frac{p-2z}{3q^2} \ , \tag{4.128}$$

and

$$c_R = c_L = -\frac{6Bp^2(p^2-q^2)}{q^2}N^2 \ . \tag{4.129}$$

Note that $B<0$. As remarked in [58], from (4.128) we see that the 2d superconformal R-symmetry involves mixing the 4d one with the baryonic symmetry. Moreover, notice that despite the mixing coefficients ϵ_2, ϵ_B and the 't Hooft anomalies being irrational numbers, this irrationality drops out of the final expression for c_R.

This result matches that of the holographic computation [56] (*c.f.* (4.104)) upon making the following identifications [58]:

$$\mathfrak{p} = p \ , \qquad \mathfrak{q} = q \ , \qquad M = BN \ . \tag{4.130}$$

Some comments are now in order. First of all, we note that since $\mathfrak{p} < \mathfrak{q}$ and $p > q$, strictly speaking this identification is *contradictory*. This issue was overlooked in the literature and certainly deserves further scrutiny in the future. Here we will not attempt to resolve it, but we will make a number of checks that confirms the plausibility of these identifications.

So far the only assumptions we made on the 2d field theories are that they are $(0,2)$ SCFTs and that their global symmetries are the same as those of the 4d parent theories. Assuming in addition that in the 2d SCFTs there exist 2d descendants of the 4d baryonic operators, we can perform some further checks. The 2d R-charges of the (naive) 2d reduction of the fields Y, Z, U_α, V_α can be computed from (4.122), namely using

$$R^{2d}[X_{2d}] = \epsilon_2 Q_{F_2}[X_{4d}] + \epsilon_B Q_B[X_{4d}] + R^{4d}[X_{4d}] \, . \tag{4.131}$$

Plugging in (4.128) and the values of the 4d charges gathered from Table 1.2 we obtain

$$\begin{aligned} R^{2d}[Y_{2d}] &= R^{2d}[Z_{2d}] = \frac{q^2 - p^2}{q^2} \ , \\ R^{2d}[U_{2d}] &= \frac{p^2}{q^2} \ , \qquad R^{2d}[V_{2d}] = 1 \ , \end{aligned} \tag{4.132}$$

in agreement with the results (4.106) for the normalised volumes of calibrated submanifolds. However, in the field theory the R-charges associated to the fields Y and Z are *negative*, indicating that a better understanding of the duality proposed in [58] would be desirable.

4.3.3 Duality Twisted $\mathcal{N} = 1$ Field Theories

In this subsection we shall extend the above computations to compactifications of the four-dimensional theories on a Riemann surface C_g, with τ varying (holomorphically) over this. In particular, we shall promote the $U(1)_D$ symmetry obtained for varying τ to be a line bundle over the Riemann surface C_g, with curvature two-form $\mathrm{d}Q$, implying that we must introduce additional terms to the 4d anomaly polynomial for this bundle.

This might seem counter-intuitive at first, since the $U(1)_D$ is not a classical symmetry of the field theory [34]. On the other hand, more abstractly one can think of anomalies as arising from non-invariance of the generating functional of current correlators, which transforms as the section of a bundle. We can then define the anomaly (polynomial) associated to the line bundle $\mathcal{L}_D$ even if $U(1)_D$ is not a classical symmetry. This is furthermore supported by the presence of the $U(1)_D$ symmetry in two separate regimes – most clearly shown for 4d $\mathcal{N}=4$ SYM: the large N limit, where the holographic dual has a $U(1)_D$ induced from the Type IIB axio-dilaton, and in the abelian theory with $N=1$ where the equations of motion are also invariant as shown in [15,16]. Clearly, further clarification of this point would be very desirable.

By introducing the additional curvature terms of the $\mathcal{L}_D$ bundle, the 4d anomaly polynomial I_6 is modified as

$$\begin{aligned} I_6^\tau =& I_6 + \frac{1}{2}k_{DIJ}c_1(\mathcal{F}_D)\wedge c_1(\mathcal{F}_I)\wedge c_1(\mathcal{F}_J) + k_{DDI}c_1(\mathcal{F}_D)\wedge c_1(\mathcal{F}_D)\wedge c_1(\mathcal{F}_I) \\ &+ k_{DDD}c_1(\mathcal{F}_D)\wedge c_1(\mathcal{F}_D)\wedge c_1(\mathcal{F}_D) - \frac{1}{24}k_D c_1(\mathcal{F}_D)\wedge p_1(T_4)\,, \end{aligned} \tag{4.133}$$

where $I\in\{R,B_I,F_I\}$ as before. The anomaly polynomial for the 2d theory, I_4^τ is again computed by integrating I_6^τ over C_g.

To get started we should now determine the additional ‘t Hooft cubic and linear anomalies involving $U(1)_D$. We shall argue that the cubic and linear anomalies involving the duality bundle will scale as N and by making a plausible assumption we will be able to compute subleading contributions to the 2d anomalies, obtaining perfect agreement with the holographic computations.

Let us consider for example the linear trace

$$k_D \;\equiv\; \mathrm{Tr}[U(1)_D] = \sum_i q_D^i\ , \tag{4.134}$$

where the sum is over all the fermions (of the 4d theory) and q_D^i are their charges under $U(1)_D$. However, exactly as for $\mathcal{N}=4$ SYM, in the non-abelian theories the bonus $U(1)_D$ is *not* a symmetry [34] and therefore these charges are not meaningful.

To circumvent this problem, it is expedient to Higgs the $\mathcal{N}=1$ quiver theories with gauge group $G=SU(N)^\chi$ to an abelian theory, at a generic point of the (mesonic) vacuum moduli space. In the low energy limit this theory has gauge group $U(1)^{N-1}$ and contains $N-1$ vector multiplets and $3N$ chiral multiplets, parameterising the flat directions of the mesonic moduli space, that is the symmetric product of N copies of the related Calabi–Yau three-fold conical singularity $\mathbb{X}=C(Y)$, $\mathrm{Sym}^N\mathbb{X}$. See [114] for some discussion in the case of the Klebanov-Witten model with $G=SU(N)^2$, and [115] for an explicit analysis in the $Y^{p,q}$ theories. This is an abelian theory for which $U(1)_D$ is now a symmetry of the equations of motion, and we can infer the charges of the fields under $U(1)_D$ from the supergravity analysis.

As we recalled in the introduction, in our conventions the supergravity Killing spinors have charge $q_D = -1/2$. In the boundary (abelian) field theory this translates to the fact that the scalar field ϕ and the fermion field ψ in a chiral multiplet have $U(1)_D$ charges satisfying $q_D[\phi]-q_D[\psi] = -1/2$. The $U(1)_D$ charge of the scalar bifundamental fields can be fixed by an extension of the arguments in [34], by noting that mesonic gauge-invariants operators (closed loops in the quiver) correspond to scalar harmonics on the Sasaki–Einstein manifold Y that are in 1–1 correspondence with holomorphic functions on the cone [116]. In particular, these modes are fluctuations of a mixture of the metric and the RR four-form potential [117]. Since these are both inert under $SL(2,\mathbb{R})$ transformations, it follows that an infinite tower of dual scalar operators is uncharged under $U(1)_D$. In $\mathcal{N} = 4$ SYM these operators are $\mathrm{Tr}X^{I_1}X^{I_2}\cdots X^{I_k}$ [24] and correspond to a KK tower on S^5, uncharged under $U(1)_D$ [118]. This clearly implies that the scalar bifundamental fields themselves must be uncharged and therefore the fermions in the chiral mutliplets have $q_D[\psi] = 1/2$. The $U(1)_D$ charge of the gauginos is fixed by the (abelian) supersymmetry transformations to be $q_D[\lambda] = 1/2$. Putting all together, we obtain

$$k_D = \frac{1}{2}3N + \frac{1}{2}(N-1) \ = \ 2N - \frac{1}{2} \quad \text{(at a generic point on the Higgs branch)}\,, \tag{4.135}$$

It remains to justify the assumption that, differently from other symmetries, for $U(1)_D$ there are no other contributions on the Higgs branch, arising from integrating out the massive off-diagonal modes [119, 120]. This is plausible, as at the origin of the Higgs branch $U(1)_D$ ceases to be a symmetry. Moreover, this scaling with N is fully consistent with the results for the $(0,4)$ theories that we discussed in [59].

Using this prescription it is straightforward to compute the mixed cubic anomaly coefficients involving one D index, k_{DIJ}. However, the result of this computation will provide for us the subleading term[17] of c_R, which we have not attempted to compute holographically, and therefore we do not present the results here. It would be interesting to compute this performing a KK analysis of the $U(1)_R$ isometry in the geometry, along the lines of [74]. Below we will discuss the matching with the holographic computations of c_R, c_L at leading order in N, and of $c_R - c_L$ at subleading order.

Universal Duality Twist: Elliptic Surface S_4^τ Case

Let us now consider the field theory dual to the solutions discussed in section 4.1.1 and compare with the results of section 4.2.2. Like the universal twist solutions revised above we shall compensate for the curvature of the base by coupling the 4d R-symmetry to a background field. This is however not sufficient to cancel off all of the curvature of C_g and we must also twist with $U(1)_D$. As before we allow the 2d R-symmetry to mix with the flavour and baryonic flavour symmetries, but we do

[17] There is no contribution to c_R from the 37 sector, therefore this is the full contribution.

not allow it to mix with $U(1)_D$, as implied by the analysis in the gravity side. The topological twist ensures that the Killing spinor equation on Σ admits a constant spinor solution. To achieve this, couple to two background fields $\mathcal{A}_i$ (unlike the constant τ cases) as

$$(\nabla_\Sigma + \mathrm{i}\mathcal{A}_1 T_D + \mathrm{i}\mathcal{A}_2 T_R)\epsilon = 0 \tag{4.136}$$

and tune these fields to cancel off the spin-connection on Σ. On Σ there is a single non-trivial component of the spin connection which satisfies

$$\mathrm{d}\omega^{12} = \mathfrak{R} = -3J - \mathrm{d}Q \ . \tag{4.137}$$

Requiring that τ is holomorphic on Σ implies that the spinor on Σ satisfies the projection condition $\gamma^{12}\epsilon = -\mathrm{i}\epsilon$ and therefore requiring that a constant spinor satisfies (4.136) implies the topological twist

$$\mathrm{d}\mathcal{A}_1 = -\mathrm{d}Q \,, \qquad \mathrm{d}\mathcal{A}_2 = 3J \,, \tag{4.138}$$

which is precisely like the topological duality twists in [18, 59] and results in the twisted $U(1)$

$$T_{\text{twist}} = T_D - \frac{1}{2}T_R^{4d} \,, \tag{4.139}$$

whilst the trial R-charge is given by

$$T_{\text{trial}}^{2d} = T_R^{4d} + \sum_I^{n_F} \epsilon_I T_I + \sum_I^{n_B} \epsilon_{B_I} T_{B_I} \ , \tag{4.140}$$

note that $U(1)_D$ does not mix in the trial R-symmetry.

Concretely the twisting induces the following identifications of the curvatures of the various bundles

$$\begin{aligned} \mathcal{F}_R^{4d} &\to \mathcal{F}_R^{2d} - \tfrac{3}{2}J_\Sigma \ , \\ \mathcal{F}_{F_I}^{4d} &\to \mathcal{F}_{F_I}^{2d} + \epsilon_I \mathcal{F}_R^{2d} \ , \\ \mathcal{F}_{B_I}^{4d} &\to \mathcal{F}_{B_I}^{2d} + \epsilon_{B_I} \mathcal{F}_R^{2d} \ , \\ \mathcal{F}_D^{4d} &\to 2\pi c_1(\mathcal{L}_D) \ , \end{aligned} \tag{4.141}$$

where F are the flavour symmetries and B the baryonic symmetries. Upon extracting the k_{RR} coefficient and extremizing with respect to the ϵ's one finds

$$\epsilon_I = 0 = \epsilon_{B_I} \,, \tag{4.142}$$

and therefore there is no mixing in 2d of the exact R-symmetry and the flavour and baryonic symmetries. This is true at leading order but may be corrected at subleading order due to cubic 't Hooft anomalies involving $U(1)_D$. The central charge is given by $c_R = 3k_{RR}$ and is obtained from reducing the I_6 on the base of the elliptic surface, Σ, as

$$c_R = \frac{16}{3}(2(g-1) + \deg(\mathcal{L}_D))a^{(4d)}\,, \tag{4.143}$$

which is in perfect agreement with (4.80). An important point to note here is that the central charge has at leading order already a τ-dependence through $\mathcal{L}_D$.

By extracting k from the I_4^τ anomaly polynomial we find the subleading contribution to be

$$(c_L - c_R)_{\text{bulk}} = -k_D \int_\Sigma c_1(\mathcal{L}_D) = -k_D \deg(\mathcal{L}_D)\ . \tag{4.144}$$

The subscript indicates that this contribution arises from the dimensional reduction of the 4d theory, ignoring the defect modes from the 7-branes. Furthermore, assuming that the contributions of the 7-branes to the spectrum are again Fermi multiplets as in [18], we can conjecture that the 3-7 defect modes gives an additional contribution

$$(c_L - c_R)_{\text{defect}} = 8N\deg(\mathcal{L}_D)\,. \tag{4.145}$$

From the discussion at the begining of this section we have $k_D = 2N - 1/2$ so that at subleading order we obtain the total contribution

$$c_L - c_R = 8N\deg(\mathcal{L}_D) - 2N\deg(\mathcal{L}_D) = 6N\deg(\mathcal{L}_D)\ , \tag{4.146}$$

which agrees with the result given in (4.82).

One may also compute the R-charges of the fields from the anomaly polynomial. As the extremization forces all the ϵ_I to vanish one finds that the R-charges of the 2d fields are the same as the R-charges of the 4d fields, in agreement with the conclusion reached in section 4.2.2.

Universal Twist: Elliptic Three-fold $\mathcal{T}_6^\tau$ Case

The field theory duals to the universal twists with an elliptic three-fold factor are obtained by a twisted reduction of the 4d $\mathcal{N} = 1$ SCFTs in section 2.4.3, whose duals are F-theoretic AdS$_5$ solutions. The field theory is reduced along a curve with constant τ, so that the standard universal twist of [58] can be implemented as in (4.123), with the trial R-symmetry given as usual. In the following we shall assume that the 4d 't Hooft coefficients still obey $k_{RRF} = k_F = 0$ and $k_{RRB} = k_B = 0$ as in the toric Sasaki–Einstein case, we make no restriction on k_{RRD} and k_D. This starting point implies that the 2d R-symmetry to leading order is given exactly by the 4d one, and we have $\epsilon_I = \epsilon_{B_i} = 0$. In particular the central charge is

$$c_R = \frac{32(g_\Sigma - 1)}{3} a_\tau^{(4d)}\,, \tag{4.147}$$

which agrees with the holographic result. Moreover the subleading contribution is given by

$$c_L - c_R = (g-1)k_R^\tau \ . \tag{4.148}$$

Since this corresponds to the twisted reduction on $\mathbb{H}^2/\Gamma$ above which the theory has no varying coupling this is the exact result to this order. In the constant τ field theory one has to subleading order $k_R = 0$ and therefore $c_L = c_R$ at subleading order. As discussed in section 4.2.3, k_R^τ is non-zero at subleading order in the varying τ field theory, and therefore non-trivial τ not only modifies the leading order central charge of the theory it also implies that the left and right moving central charges differ at subleading order.

As a final check of our results in section 4.2.3 the identification of the 2d R-symmetry with the 4d one implies that the R-charges of the fields in 2d and 4d are identical, which agrees with the results presented in the holographic setup.

Baryonic Duality Twist

We now discuss theories with varying coupling, which have a baryonic symmetry. We can compactify on a complex curve C_g of genus $g \neq 1$ and preserve supersymmetry by twisting with $U(1)_D$, as explained in section 4.3.1. As the supercharges are uncharged under the baryonic (and flavour) symmetries we are free to twist with these as well. In particular, we take $C_0 = \mathbb{P}^1$, with curvature given by $-\mathrm{d}Q$, which is also the connection of the duality line bundle $\mathcal{L}_D$. Concretely, the topological twist we take is

$$T_{\text{twist}} = BT_B + T_D \ . \tag{4.149}$$

We again assume that the R-symmetry does not mix with $U(1)_D$ and therefore we take as trial R-charge

$$T_{\text{trial}} = \epsilon_2 T_2 + \epsilon_B T_B + T_R^{4d} \ . \tag{4.150}$$

Under the twisting the curvatures of the various bundles become[18]

$$\begin{aligned}
&\mathcal{F}_R^{4d} \to \mathcal{F}_R^{2d} \ , \\
&\mathcal{F}_{F_1}^{4d} \to \mathcal{F}_{F_1}^{2d} \ , \\
&\mathcal{F}_{F_2}^{4d} \to \mathcal{F}_{F_2}^{2d} + \epsilon_2 \mathcal{F}_R^{2d} \ , \\
&\mathcal{F}_B^{4d} \to \mathcal{F}_B^{2d} + \epsilon_B \mathcal{F}_R^{2d} - B t_g \ , \\
&\mathcal{F}_D^{4d} \to 2\pi c_1(\mathbb{P}^1) \ .
\end{aligned} \tag{4.151}$$

The last line is fixed as the compactification geometry is an elliptic K3 surface. The anomaly polynomial for the 2d theory, I_4^τ is computed by integrating I_6^τ in (4.133)

[18] Note that there is a minus sign difference in the $\mathcal{F}_B$ term with that in equation (2.47) of [58]. We fixed this by first recovering the results for constant τ on a T^2 via the anomaly polynomial.

over the base $\mathbb{P}^1$ of the elliptic K3$^\tau$[19]

$$\begin{aligned}\int_{\mathbb{P}^1} I_6^\tau = I_4^\tau \supset &-(B(k_{R1B}+k_{12B}\epsilon_2+k_{1BB}\epsilon_B)+2(k_{R1D}+k_{12D}\epsilon_2+k_{1BD}\epsilon_B))c_1(\mathcal{F}_1)\wedge c_1(\mathcal{F}_R)\\
&-(B(k_{R2B}+k_{22B}\epsilon_2+k_{2BB}\epsilon_B)+2(k_{R2D}+k_{22D}\epsilon_2+k_{2BD}\epsilon_B))c_1(\mathcal{F}_2)\wedge c_1(\mathcal{F}_R)\\
&-(B(k_{RBB}+k_{2BB}\epsilon_2+k_{BBB}\epsilon_B)+2(k_{RBD}+k_{2BD}\epsilon_2+k_{BBD}\epsilon_B))c_1(\mathcal{F}_B)\wedge c_1(\mathcal{F}_R)\\
&-\frac{1}{2}\left[B\{k_{RRB}+\epsilon_B(2k_{RBB}+k_{BBB}\epsilon_B)+\epsilon_2(2k_{RRB}+k_{22B}\epsilon_2+2k_{2BD}\epsilon_B)\}\right.\\
&\left.+2\{k_{RRD}+\epsilon_B(2k_{RBD}+k_{BBD}\epsilon_B)+\epsilon_2(2k_{R2D}+k_{22D}\epsilon_2+2k_{2BD}\epsilon_B)\}\right]c_1(\mathcal{F}_R)^2\\
&+\frac{1}{24}(Bk_B+2k_D)p_1(T_2)\end{aligned} \tag{4.152}$$

Comparing this with the general structure of the I_4 polynomial (4.117) and (4.118) yields

$$\begin{aligned}c_R = 3k_{RR} = &-3\left[B\{k_{RRB}+\epsilon_B(2k_{RBB}+k_{BBB}\epsilon_B)+\epsilon_2(2k_{RRB}+k_{22B}\epsilon_2+2k_{2BD}\epsilon_B)\}\right.\\
&\left.+2\{k_{RRD}+\epsilon_B(2k_{RBD}+k_{BBD}\epsilon_B)+\epsilon_2(2k_{R2D}+k_{22D}\epsilon_2+2k_{2BD}\epsilon_B)\}\right]\,,\\
c_L-c_R = &-Bk_B-2k_D\,.\end{aligned} \tag{4.153}$$

The exact central charge is obtained by extremizing c_R with respect to ϵ_B, ϵ_2, the expression one obtains is prohibitively large and so we do not present it here. The key is to note how the various 't Hooft anomalies scale with N [108].Those not involving the duality symmetry, $U(1)_D$ will be unaffected by its inclusion and scale as N^2, on the other hand any term involving $U(1)_D$ will scale as N and therefore it will be subleading. Observe that in the universal twist solutions presented previously a non-trivial variation induces a shift in the central charge at leading order, not just at subleading order as is the present case.

Note that so far we have not specified a theory, and therefore the conclusion that the leading order central charge is unchanged with respect to the value of the same theory, compactified on $C_{g=1}=T^2$, and twisted by $T_{\text{twist}}=BT_B$ is quite general. Specialising to the $Y^{p,q}$ quivers, we of course recover the result (4.129)

$$c_R = -\frac{6Bp^2\,(p^2-q^2)}{q^2}N^2+O(N)\,. \tag{4.154}$$

This is in agreement with our observations from gravity that the corrections due to τ are subleading in N. Using $k_B=0$ we also obtain

$$(c_L-c_R)_{\text{bulk}} = -4N+1\,, \tag{4.155}$$

where again the subscript indicates that this contribution arises from the dimensional

[19]As the expression one finds for I_4 is unwieldy we present only the salient terms.

reduction of the 4d theory, ignoring the defect modes from the 7-branes. For an elliptic K3 the 3-7 defect modes gives an additional contribution

$$(c_L - c_R)_{\text{defect}} = 16N \,, \tag{4.156}$$

so that the total contribution at order $O(N)$ is precisely $12N$ as in (4.105).

4.4 Concluding remarks

For duals to 2d $(0,2)$ SCFTs we discussed two classes of solutions, which are all based on the general form of the F-theory solution (i.e. including the axio-dilaton into the geometric description in terms of the elliptic fibrations) given by

$$\text{AdS}_3 \times (S^1 \to \mathcal{Y}_8^\tau) \,. \tag{4.157}$$

Here $\mathcal{Y}_8^\tau$ is elliptically fibered. The base of this elliptic fibration $\widetilde{\mathcal{M}}_6$ is a Kähler three-fold. The first class of solutions are of the type $\widetilde{\mathcal{M}}_6 = \Sigma \times \mathcal{M}_4$, i.e. a product of a curve and a surface. This gives rise to the universal twist solutions, where the elliptic fibration is non-trivial only over one of the two factors. The key characteristic of these universal twist solutions in F-theory is that they do not have any Calabi-Yau factors, i.e. the elliptic fibration restricted to Σ and $\mathcal{M}_4$, respectively, cannot be Ricci flat! The second class of solutions is obtained by imposing that there is explicitly a Ricci-flat factor in the direct product $\mathcal{Y}_8^\tau = \mathcal{M}_4 \times \text{K3}^\tau$. The resulting solutions are of the type $\text{AdS}_3 \times \text{K3}^\tau \times \mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$, or as Type IIB solution $\text{AdS}_3 \times \mathbb{P}^1 \times \mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$, where $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ are circle-fibrations over $\mathbb{F}_0$. These are the baryonic twist solutions. In each case we determined the holographic central charges and matched them to dual field theory, where the central charge is obtained using c-extremization applied in the context of 4d $\mathcal{N} = 1$ field theories with varying coupling. Key to our analysis are various topological twists of the 4d theories that involve the $U(1)_D$ "bonus" symmetry inherited from Type IIB supergravity. In particular, we have demonstrated in several examples how this twisting affects the F-theory geometry as well as the dual field theories, through an analysis based on an $U(1)_D$-augmented anomaly polynomial of these theories.

For the baryonic twist solutions, based on the $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ geometries, we have uncovered some puzzling aspects (see Section (4.3.2)) of the proposed duality with the $Y^{p,q}$ quiver gauge theories [58], already present in the solutions with constant τ. It is clearly an interesting question to resolve these puzzles, and we hope to return to this in the near future.

Finally, in this section we have shown that a simple extension of the anomaly polynomial to the "bonus" $U(1)_D$ symmetry provides a powerful tool for studying field theories with varying couplings. Work following this observation was undertaken in [108] to make the arguments that we employed in Section 4.3 more rigorous and to deduce the contribution of the seven-brane modes to the central charges of the two-dimensional theories.

Chapter 5

AdS_5

During our investigation of AdS_3 solutions with varying axio-dilaton we stumbled upon AdS_5 solutions with varying axio-dilaton, (section 2.4.3) that had not been appreciated previously. In [7] Type IIB solutions with AdS_5 factors and an identity structure were classified with general fluxes. These may be viewed as F-theoretic as τ is allowed to vary in their equations. The case of vanishing five-form flux was not considered however, and was implicitly assumed to be non-vanishing throughout. Attempts to set $F_5 = 0$ later leads to inconsistencies due to division by zero. In [1] two new supersymmetric solutions were found with $F_5 = 0$ and were the first of their type. To obtain these solutions the authors began with two well known AdS_5 Sasaki-Einstein solutions and perform a Non-Abelian T-duality (NATD) on an $SU(2)$ isometry to IIA followed by a T-duality along a remaining $U(1)$ to return to IIB. The supersymmetric solutions that are obtained have seed solutions $\mathrm{AdS}_5 \times T^{(1,1)}$ and $\mathrm{AdS}_5 \times Y^{p,q}$. Unfortunately these new solutions are singular and it was hoped that by completing this classification we would be able to find new non-singular solutions of this form. One can also perform two T-dualities in the spirit of the above NATD procedure. The resulting solutions have $F_5 = 0$ and can be viewed as an infinite β limit of a beta deformation [121]. Again these solutions have singularities which are located at the vertices of the associated toric polytope, we give an example of these solutions in section 5.4. Finding non-singular AdS_5 solutions with vanishing F_5 remains an open problem.

We shall again keep τ arbitrary. Unlike in the AdS_3 case presented above we will not be able to give an F-theoretic interpretation in terms of an auxillary elliptic fibration. Instead the one-form P introduced in section 1.1.1 will appear non-trivially in various bilinear equations.

We shall use the method of G-structure techniques as before. We find that the internal manifold admits an identity structure which allows us to determine the metric and three-form flux completely. The geometry includes a hypersurface-orthogonal Killing vector which is a symmetry of the full solution and corresponds to the $U(1)_R$ R-symmetry in the putative dual SCFT. Supersymmetry implies all the Bianchi identities and equation's of motion, including all components of the Einstein equation, similar results are true in [7], though the techniques to show this there do not work in the $F_5 = 0$ case. We shall present a new singular solution and show how one of the solutions in [1] fits into the classification. Some technical material is relegated to appendix D

The content of this section is taken from [122] and from unpublished notes with N. Macpherson and D. Martelli.

5.1 The conditions for supersymmetry in $d = 5$

We shall follow the conventions and notation of [7] for the Type IIB supergravity field content, equations of motion, and supersymmetry variations, see section 1.1.1 for further details.

We wish to characterise the most general class of bosonic supersymmetric solutions of Type IIB supergravity with $SO(4,2)$ symmetry and *vanishing five-form flux*. Namely we require that

$$F_5 = 0 \ , \tag{5.1}$$

which means that the solutions we study correspond to configurations *without D3 branes*. This is a slight difference to the analysis performed in [7], where it was (implicitly) assumed throughout that $F_5 \neq 0$. As pointed out in the introduction it is not possible to simply set $F_5 = 0$ in the final equations presented in [7]. Nevertheless much of the initial analysis conducted in their paper can be utilised and we shall indicate when this is possible and when it is not.

The $d = 10$ metric, in Einstein frame, takes the form of a warped product

$$\mathrm{d}s^2_{10} = \mathrm{e}^{2\Delta}\left(\mathrm{d}s^2(\mathrm{AdS}_5) + \mathrm{d}s^2(\mathcal{M}_5)\right) , \tag{5.2}$$

where $\mathrm{d}s^2(\mathrm{AdS}_5)$ is the metric on AdS_5 with Ricci tensor given by $R_{\mu\nu} = -4m^2(g_{\mathrm{AdS}_5})_{\mu\nu}$ and $\mathrm{d}s^2(\mathcal{M}_5)$ is the metric on a five-dimensional Riemannian internal space $\mathcal{M}_5$. In order to preserve the $SO(4,2)$ symmetry of the metric we require the fields to take values in; $\Delta \in \Omega^0(\mathcal{M}_5,\mathbb{R})$, $P \in \Omega^1(\mathcal{M}_5,\mathbb{C}), Q \in \Omega^1(\mathcal{M}_5,\mathbb{R})$ and $G \in \Omega^3(\mathcal{M}_5,\mathbb{C})$. Notice that with this ansatz the Bianchi identity for F_5 is trivially satisfied and it is therefore consistent to set $F_5 = 0$ without imposing any further conditions.

We will use the most general ansatz for the Killing spinor consistent with preserving minimal supersymmetry in AdS_5. This takes the form

$$\epsilon = \mathrm{e}^{\Delta/2}(\psi \otimes \xi_1 \otimes \theta + \psi^c \otimes \xi_2^c \otimes \theta) \ , \tag{5.3}$$

where we have rescaled the spinor by the factor $\mathrm{e}^{\Delta/2}$ for later convenience. Here ψ is a Killing spinor on AdS_5 and ξ_i are two independent $Spin(5)$ spinors on M_5.

Further discussion about the spinor ansatz and conventions can be found in appendix A of [7]. Requiring supersymmetry to be preserved yields the following conditions

$$\mathcal{D}_m\xi_1 + \frac{1}{8}\mathrm{e}^{-2\Delta}\gamma^{m_1m_2}G_{mm_1m_2}\xi_2 - \frac{\mathrm{i}}{2}m\gamma_m\xi_1 = 0 \ , \tag{5.4}$$

$$\bar{\mathcal{D}}_m\xi_2 + \frac{1}{8}\mathrm{e}^{-2\Delta}\gamma^{m_1m_2}G^*_{mm_1m_2}\xi_1 - \frac{\mathrm{i}}{2}m\gamma_m\xi_2 = 0 \ , \tag{5.5}$$

$$\gamma^m\partial_m\Delta\xi_1 + \mathrm{i}m\xi_1 - \frac{1}{48}\mathrm{e}^{-2\Delta}G_{m_1..m_3}\gamma^{m_1..m_3}\xi_2 = 0 \ , \tag{5.6}$$

$$\gamma^m\partial_m\Delta\xi_2 + \mathrm{i}m\xi_2 - \frac{1}{48}\mathrm{e}^{-2\Delta}G^*_{m_1..m_3}\gamma^{m_1..m_3}\xi_1 = 0 \ , \tag{5.7}$$

$$P_m\gamma^m\xi_2 + \frac{1}{24}\mathrm{e}^{-2\Delta}\gamma^{m_1..m_3}G_{m_1..m_3}\xi_1 = 0 \ , \tag{5.8}$$

$$P^*_m\gamma^m\xi_1 + \frac{1}{24}\mathrm{e}^{-2\Delta}\gamma^{m_1..m_3}G^*_{m_1..m_3}\xi_2 = 0 \ . \tag{5.9}$$

These can be obtained straightforwardly from the equations (3.3) - (3.8) in [7], by setting $f = 0$[1].

Special cases

The possible stabilizer groups of the Spin(5) spinors ξ_i are the identity group or $SU(2)$. Consequently M_5 may admit either an identity structure or an $SU(2)$ structure.

Let us first consider the case of an $SU(2)$ structure. This corresponds to setting one of the spinors to zero, without loss of generality, let us assume $\xi_2 = 0$. Then equation (5.6) reads

$$\gamma^m\partial_m\Delta\xi_1 = -\mathrm{i}m\xi_1 \ . \tag{5.10}$$

Following the use of Clifford algebra identities one can show easily that $\partial_n\Delta = 0$, and inserting this back into (5.10) we reach the contradiction $m\xi_1 = 0$. Whilst the $F_5 \neq 0$ case allowed for an $SU(2)$ structure on M_5, comprising the well known Sasaki-Einstein solutions, we conclude that there are no supersymmetric $\mathrm{AdS}_5 \times M_5$ solutions with $F_5 = 0$ in Type IIB supergravity with M_5 admitting an $SU(2)$ structure[2].

Another interesting case to consider is $G = 0$. Such putative solutions would arise purely from D7 branes, and would be motivated by F-theory constructions. Setting $G = 0$ in equation (5.6) and (5.7) once again gives (5.10) and an analogous equation for ξ_2 which implies $\xi_1 = 0 = \xi_2$ and hence no supersymmetry is preserved. We therefore conclude that supersymmetric AdS_5 solutions of Type IIB supergravity with vanishing five-form *and* three-form fluxes do not exist.

In the remainder of the section we will assume that G is non-vanishing, and that both spinors ξ_i are not identically zero, thus giving a (local) identity structure on

[1] f is the constant defined in [7] as $F_5 = f(\mathrm{dvol}(\mathrm{AdS}_5) + \mathrm{dvol}(\mathcal{M}_5))$.

[2] In [123] it has also been shown that in type IIA supergravity there are no solutions of the form $\mathrm{AdS}_5 \times \mathcal{M}_5$ with $\mathcal{M}_5$ having an $SU(2)$ structure either.

$\mathcal{M}_5$.

5.2 Bilinear equations

The identity structure is characterised by a set of one-forms, constructed as spinor bilinears, that can be used to define a canonical orthonormal frame on $\mathcal{M}_5$. In the analysis of the algebraic and differential conditions equivalent to the supersymmetry equations it is useful to consider also a number of scalar and two-form bilinears. We define these following the notation in [7] and we list them in appendix D.1. From the algebraic condition (3.25) in [7] we see that $F_5 = 0$ implies that $\sin\zeta = 0$[3]; we can therefore import the bilinear equations from [7] where we set $\sin\zeta \equiv 0$ and $f \equiv 0$. The resulting differential conditions are[4]

$$
\begin{aligned}
\mathrm{e}^{-4\Delta}\mathrm{d}(\mathrm{e}^{4\Delta}S) &= 3\mathrm{i}mK \ , && (5.11)\\
\mathrm{e}^{-6\Delta}\mathcal{D}(\mathrm{e}^{6\Delta}K_3) &= P\wedge K_3^* - 4\mathrm{i}mW - \mathrm{e}^{-2\Delta} * G \ , && (5.12)\\
\mathrm{e}^{-4\Delta}\mathrm{d}(\mathrm{e}^{4\Delta}K_4) &= -2mV \ , && (5.13)\\
\mathrm{e}^{-8\Delta}\mathrm{d}(\mathrm{e}^{8\Delta}K_5) &= -6mU \ , && (5.14)
\end{aligned}
$$

while the algebraic conditions are

$$
\begin{aligned}
Z = 0 &= \sin\zeta, \quad A = 1 \ , && (5.15)\\
2i_{K_3}\mathrm{d}\Delta &= i_{K_3^*}P \ , && (5.16)\\
i_{K_5}\mathrm{d}\Delta = 0 &= i_{K_5}P \ , && (5.17)\\
(1-|S|^2)\mathrm{e}^{-2\Delta} * G &= 2P\wedge K_3^* - (4\mathrm{d}\Delta + 4\mathrm{i}mK_4)\wedge K_3 \\
&\quad +2 * (P\wedge K_3^*\wedge K_5 - 2\mathrm{d}\Delta\wedge K_3\wedge K_5) \ . && (5.18)
\end{aligned}
$$

Note that in [7] the differential condition on K_4 was implied by the remaining ones, because this one-form could be expressed as a linear combination of the other bilinears, as can be seen from (D.4), however this is no longer the case. Indeed, more generally, the orthonormal frame that we will use here, differs from the analogous one introduced in [7]. Using this orthonormal frame, presented in appendix D.1, we find that the metric takes the form

$$
\mathrm{d}s^2(\mathcal{M}_5) = \frac{K_5^2}{|S|^2} + \frac{K_4^2}{1-|S|^2} + \frac{K_3\otimes K_3^*}{1-|S|^2} + \frac{|S|^2}{1-|S|^2}(\mathfrak{Im}[S^{-1}K])^2 \ . \qquad (5.19)
$$

[3]Following the argument in appendix C of [7], and imposing $\sin\zeta = 0$, we find that it is not possible to have the spinors ξ_i non-vanishing and linearly dependent. We therefore restrict to the case of them being independent and admitting an identity structure.

[4]Here and in the rest of the section $*$ denotes the Hodge star operator with respect to the five-dimensional metric $\mathrm{d}s^2(\mathcal{M}_5)$.

This should be contrasted with the metric written in equation (3.53) of [7].

It is immediate from the analysis of [7] that K_5 defines a Killing vector. Moreover, here we will find that additionally K_5 is in fact a hypersurface-orthogonal Killing vector. This is most easily seen after we introduce local coordinates in the following section.

Analogously to [7], one can show K_5 is in fact a symmetry of the full solution, namely

$$\begin{aligned} \mathcal{L}_{K_5}\Delta &= \mathcal{L}_{K_5}\Phi = \mathcal{L}_{K_5}C^{(0)} = 0 \ , \\ \mathcal{L}_{K_5}G &= 0 \ . \end{aligned} \tag{5.20}$$

In a putative dual $d = 4$ superconformal field theory this corresponds to having $U(1)$ R-symmetry and hence $\mathcal{N} = 1$ supersymmetry.

Let us now show that supersymmetry implies that all the equations of motion and Bianchi identities are satisfied. Most of the arguments presented in [7] to show that all the equations of motion and the P Bianchi identity are implied by supersymmetry can be used in our case, however, as alluded to in the introduction the argument showing that the Bianchi identity for G is satisfied is not valid if $F_5 = 0$. Below we present an argument that applies to both cases. Using the supersymmetry equations, we find

$$\mathcal{D}(\mathrm{e}^{6\Delta}X) = \mathrm{e}^{6\Delta}(3\mathrm{i}m * X - \mathrm{e}^{-2\Delta}SG + P \wedge Y) \ , \tag{5.21}$$

$$\mathrm{e}^{-6\Delta}\bar{\mathcal{D}}(\mathrm{e}^{6\Delta}Y) = 3\mathrm{i}m * Y + \mathrm{e}^{-2\Delta}SG^* + P^* \wedge X \ , \tag{5.22}$$

$$\mathrm{e}^{-6\Delta}\mathcal{D}(\mathrm{e}^{6\Delta} * X) = -\mathrm{e}^{-2\Delta}G \wedge K + P \wedge *Y \ . \tag{5.23}$$

These equations are true even including a non-zero F_5, as this drops out of the expressions. To recover the Bianchi identity for G one should take $\mathcal{D}$ of (5.21) and use (5.11), (5.22) and (5.23). As in [7], we conclude:

For the class of solutions with metric of the form (5.2), vanishing five-form flux and fluxes respecting $SO(4,2)$ symmetry, all the equations of motion and Bianchi identities are implied by supersymmetry.

5.3 Introducing local coordinates

In this section we shall introduce local coordinates in which the set of BPS equations become more explicit. We begin by reducing on the Killing direction defined by K_5, resulting in a 4-1 splitting of the metric. The transverse four-dimensional metric to the Killing direction admits an integrable almost product structure giving a further 3-1 splitting. The resulting BPS equations take a similar form to those presented in [7] in the $F_5 \neq 0$ case, but they are different. We shall conclude this section by introducing explicit coordinates on the remaining three-dimensional part of the metric, and obtaining expressions for the NS-NS and R-R two-form potentials.

We begin by choosing a local coordinate adapted to the Killing direction defined by K_5. As a vector we have

$$K_5^{\#} = 3m\frac{\partial}{\partial\psi} \ , \tag{5.24}$$

and as a one-form

$$K_5 = \frac{|S|^2}{3m}(\mathrm{d}\psi + \rho) \ , \tag{5.25}$$

where ρ is a one-form with no $\mathrm{d}\psi$ term. The factor of $3m$ is chosen for later convenience. The Lie derivative of S with respect to $K_5^{\#}$ is

$$\mathcal{L}_{K_5^{\#}} S = -3\mathrm{i}mS \ , \tag{5.26}$$

from which we find

$$S = -|S|\mathrm{e}^{-i\psi} \ . \tag{5.27}$$

It is convenient to make the redefinitions

$$\mu = \mathrm{e}^{-4\Delta} \ , \quad \eta = \mathrm{e}^{4\Delta}|S| \ . \tag{5.28}$$

Then from (5.11) we have

$$K = \frac{\mu\ \mathrm{e}^{-\mathrm{i}\psi}}{3m}(\eta\mathrm{d}\psi + \mathrm{i}\mathrm{d}\eta) \ , \tag{5.29}$$

and using the expression for K in appendix D.1 we deduce that

$$K_5 = \frac{\eta^2\mu^2}{3m}\mathrm{d}\psi \ , \tag{5.30}$$

and is therefore a hypersurface-orthogonal Killing vector. Notice that the Killing vector is not fibered, $\rho = 0$, and this differs from [7]. Making the additional redefinitions

$$K_3 = \frac{\mu^{3/2}}{3m}\sigma \ , \quad K_4 = \frac{\mu}{3m}\beta \ , \tag{5.31}$$

the metric becomes

$$9m^2\mathrm{d}s^2(\mathcal{M}_5) = \frac{1}{1-\eta^2\mu^2}\left(\mu^3\sigma\otimes\sigma^* + \mu^2\beta^2 + \mu^2\mathrm{d}\eta^2\right) + \eta^2\mu^2\mathrm{d}\psi^2 \ . \tag{5.32}$$

Here β is a real one-form and σ is a complex one-form, and both have no leg along the Killing direction. We should now re-express the differential and algebraic conditions in terms of these redefined quantities. We find that (5.14) is automatically satisfied, whilst equation (5.13) becomes

$$\mathrm{d}\beta = \frac{\mu^2}{3(1-\eta^2\mu^2)}\left[\mathrm{i}\sigma^* \wedge \sigma - 2\eta \mathrm{d}\eta \wedge \beta\right] \ . \tag{5.33}$$

Equation (5.12) becomes

$$\begin{aligned}\mathcal{D}\sigma = \frac{1}{\eta^2\mu^2-1}\Bigg[&(1+\eta^2\mu^2)P\wedge\sigma^* + \frac{4\mu^2\eta}{3}\mathrm{d}\eta\wedge\sigma + \mathrm{d}\ln\mu\wedge\sigma \\ &+\frac{\eta^2\mu^2}{3m} * \left(2P\wedge\sigma^*\wedge\mathrm{d}\psi + \mathrm{d}\ln\mu\wedge\sigma\wedge\mathrm{d}\psi\right)\Bigg] \ ,\end{aligned} \tag{5.34}$$

where we have used the expression for $*G$ given in (5.12). The remaining algebraic equations read

$$2i_{\sigma^*}P = -i_\sigma \mathrm{d}\ln\mu \ , \tag{5.35}$$

$$\mathcal{L}_{\frac{\partial}{\partial\psi}}\mu = \mathcal{L}_{\frac{\partial}{\partial\psi}}\Phi = \mathcal{L}_{\frac{\partial}{\partial\psi}}C^0 = 0 \ . \tag{5.36}$$

These constitute the set of necessary and sufficient conditions that one needs to satisfy for supersymmetry.

To make these equations completely explicit, we can introduce the four remaining coordinates. It is a standard calculation (for example starting with (5.29)) to check that the four-dimensional metric transverse to the Killing direction has an integrable almost product structure. This allows one to introduce "splitting coordinates", and gives a 3-1 splitting of the metric. In these coordinates the metric still takes the form presented in (5.32) however now the one-forms β and σ have no $\mathrm{d}\eta$ term, though they are still in general functions of η. We may then split the five-dimensional exterior derivative as

$$\mathrm{d} = \mathrm{d}_3 + \mathrm{d}\eta\frac{\partial}{\partial\eta} + \mathrm{d}\psi\frac{\partial}{\partial\psi} \ , \tag{5.37}$$

where d_3 is the exterior derivative on the three-dimensional metric defined by the integrable almost product structure. Equation (5.33) now reads

$$\mathrm{d}_3\beta = \frac{\mathrm{i}\mu^2}{3(1-\eta^2\mu^2)}\sigma^*\wedge\sigma \ , \tag{5.38}$$

$$\partial_\eta\beta = -\frac{2\eta\mu^2}{3(1-\eta^2\mu^2)}\beta \ , \tag{5.39}$$

whilst (5.34) reads[5]

$$\begin{aligned}
\mathrm{d}_3\sigma - \mathrm{i}Q_3\wedge\sigma &= \frac{1}{\eta^2\mu^2-1}\Big[(1+\eta^2\mu^2)P_3\wedge\sigma^* + \mathrm{d}_3\ln\mu\wedge\sigma \\
&\quad -3m\eta\sqrt{1-\eta^2\mu^2}\,*_3\left(2P_\eta\sigma^*+\partial_\eta\ln\mu\sigma\right)\Big]\ , \qquad (5.40)\\
\partial_\eta\sigma - \mathrm{i}Q_\eta\sigma &= \frac{1}{\eta^2\mu^2-1}\Big[(1+\eta^2\mu^2)P_\eta\sigma^* + \frac{4\mu^2\eta}{3}\sigma + \partial_\eta\ln\mu\ \sigma \\
&\quad -\frac{\mu^2\eta}{3m\sqrt{1-\eta^2\mu^2}}\,*_3\left(2P_3\wedge\sigma^*+\mathrm{d}_3\ln\mu\wedge\sigma\right)\Big]\ , \qquad (5.41)
\end{aligned}$$

where we have used (5.36).

Thus for the most general, minimally supersymmetric AdS_5 solutions with vanishing five-form flux we need to solve the four differential equations (5.38) - (5.41) subject to the algebraic equation (5.35). We note that the integrability equation for (5.38) and (5.39) is automatically satisfied upon using (5.35), (5.40) and (5.41).

We may now introduce the three remaining coordinates along β and σ, which we will denote as x and y_i, with $i=1,2$. In particular, we write the three independent real one-forms as

$$\begin{aligned}
\beta &= \gamma_x\mathrm{d}x + \gamma_{y_1}\mathrm{d}y_1 + \gamma_{y_2}\mathrm{d}y_2\ ,\\
\mathrm{Re}\,[\sigma] &= \rho_x\mathrm{d}x + \rho_{y_1}\mathrm{d}y_1 + \rho_{y_2}\mathrm{d}y_2\ , \qquad (5.42)\\
\mathrm{Im}\,[\sigma] &= \kappa_x\mathrm{d}x + \kappa_{y_1}\mathrm{d}y_1 + \kappa_{y_2}\mathrm{d}y_2\ .
\end{aligned}$$

Notice that generically we cannot simplify further these expressions, and the equations (5.38) - (5.41) take the form of a very complicated set of coupled PDE's. An explicit example of a rather generic solution will be presented later in section 5.5.

To obtain the explicit form of the NS-NS two-form B and the R-R two-form $C^{(2)}$ we can combine equations (5.21) and (5.22), to obtain

$$\mathcal{D}(\mathrm{e}^{6\Delta}(Y^*-X)) = -3\mathrm{i}m\mathrm{e}^{6\Delta}*(Y^*+X) + \mathrm{e}^{4\Delta}(S+S^*)G + \mathrm{e}^{6\Delta}P\wedge(X^*-Y) \qquad (5.43)$$

It is then simple, but tedious, to extract the two two-forms B and $C^{(2)}$ from the real and imaginary parts of this equation, by using (5.11) - (5.14) and the results of appendix D.1. We find

$$\begin{aligned}
B-\omega_B &= \frac{\mathrm{e}^{\Phi/2}\mu}{9m^2}\mathrm{Re}\,[\sigma]\wedge\mathrm{d}\psi\ , \qquad (5.44)\\
C^{(2)}-\omega_C &= C^{(0)}\frac{\mathrm{e}^{\Phi/2}\mu}{9m^2}\mathrm{Re}\,[\sigma]\wedge\mathrm{d}\psi + \frac{\mathrm{e}^{-\Phi/2}\mu}{9m^2}\mathrm{Im}\,[\sigma]\wedge\mathrm{d}\psi\ , \qquad (5.45)
\end{aligned}$$

where ω_B and ω_C are undetermined closed two-forms. Analogous expressions rele-

[5] Here $*_3$ is the hodge star on the three-dimensional metric defined by the integrable almost product structure.

vant for the $F_5 \neq 0$ case were given in [124].

5.4 A new solution

In this section we perform two T-dualities on the base of the conifold, referred thereafter as Romans' solution [125], consistent with preserving supersymmetry. The dual field theory to the AdS$_5$ supergravity solution with internal manifold the base of the conifold was give in [126]. Romans' solution is given by

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}s^2(\mathrm{AdS}_5) + \frac{1}{m^2}\big(\lambda_1^2(\mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\varphi_1^2 + \mathrm{d}\theta_2^2 + \sin^2\theta_2 \mathrm{d}\varphi_2^2) \\
&\quad + \lambda^2(\mathrm{d}\psi + \cos\theta_1 \mathrm{d}\varphi_1 + \cos\theta_2 \mathrm{d}\varphi_2)^2\big) \ , \\
F_5 &= 4m(1+*)\mathrm{vol}(\mathrm{AdS}_5) \ ,
\end{aligned} \tag{5.46}$$

with all other fluxes vanishing and with constant dilaton. We are using the metric on AdS$_5$ which has Ricci-tensor given by $R_{\mu\nu} = 4m^2 g_{\mu\nu}$. The two constants have values $\lambda = \frac{1}{3}$ and $\lambda_1 = \frac{1}{\sqrt{6}}$. The internal space is the coset space $SU(2) \times SU(2)/U(1)$, in particular both φ_1 and φ_2 define Killing directions and we may T-dualize along them without breaking supersymmetry.

5.4.1 T-dual on the φ_1 direction

We preform a T-duality along the Killing direction defined by φ_1. For simplicity define the function

$$W = \lambda_1^2 \sin^2\theta_1 + \lambda^2 \cos^2\theta_1 \ . \tag{5.47}$$

After performing the T-dual the solution is:

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}s^2(\mathrm{AdS}_5) + \frac{1}{m^2}\Bigg(\lambda_1^2(\mathrm{d}\theta_1^2 + \mathrm{d}\theta_2^2 + \sin^2\theta_2 \mathrm{d}\varphi_2^2) \\
&\quad + \frac{\lambda^2\lambda_1^2 \sin^2\theta_1}{W}(\mathrm{d}\psi + \cos\theta_2 \mathrm{d}\varphi_2)^2 + \frac{1}{W}\mathrm{d}\varphi_1^2\Bigg) \ , \\
B &= \frac{\lambda^2 \cos\theta_1}{m^2 W}(\mathrm{d}\psi + \cos\theta_2 \mathrm{d}\varphi_2) \wedge \mathrm{d}\varphi_1 \ , \\
F_4 &= \frac{4\lambda\lambda_1^4 \sin\theta_1 \sin\theta_2}{m^3} \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \wedge \mathrm{d}\varphi_2 \wedge \mathrm{d}\psi \\
\mathrm{e}^{-2\Phi_{IIA}} &= W
\end{aligned} \tag{5.48}$$

5.4.2 T-dual of the Type IIA solution along the φ_2 direction

Next perform a further T-duality along the Killing direction ∂_{φ_2}. After defining the function

$$\mathrm{U} = \sin^2\theta_2 W + \lambda^2 \sin^2\theta_1 \cos^2\theta_2 \ , \tag{5.49}$$

the double T-dual solution becomes

$$
\begin{aligned}
\mathrm{d}s^2(IIB) &= \mathrm{d}s^2(\mathrm{AdS}_5) + \frac{1}{m^2 W}\mathrm{d}\varphi_1^2 + \frac{\lambda_1^2}{m^2}\left(\mathrm{d}\theta_1^2 + \mathrm{d}\theta_2^2 + \frac{\lambda^2 \sin^2\theta_1 \sin^2\theta_2}{\mathrm{U}}\mathrm{d}\psi^2\right) \\
&\quad + \frac{W}{m^2\lambda_1^2\mathrm{U}}\left(\mathrm{d}\varphi_2 - \frac{\lambda^2\cos\theta_1\cos\theta_2}{W}\mathrm{d}\varphi_1\right)^2 , \qquad (5.50)\\
F_3 &= -\frac{4\lambda\lambda_1^4}{m^2}\sin\theta_1\sin\theta_2 \mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\psi , \qquad (5.51)\\
B &= \frac{\lambda^2\sin^2\theta_1\cos\theta_2}{m^2\mathrm{U}}\mathrm{d}\psi\wedge\mathrm{d}\varphi_2 + \frac{\lambda^2\sin^2\theta_2\cos\theta_1}{m^2\mathrm{U}}\mathrm{d}\psi\wedge\mathrm{d}\varphi_1 , \qquad (5.52)\\
e^{-2\Phi_{IIB}} &= \lambda_1^2\mathrm{U} \qquad (5.53)
\end{aligned}
$$

Notice that the symmetry of exchanging φ_1 and φ_2 is preserved after the two T-duals as it should be. Furthermore the string coupling-constant from the original solution gets mapped to $g_s \to g_s m^2 {\ell_s}^2$ under the two T-dualities.

The metric has curvature singularities at four points, $\{\theta_1 = \theta_2 = 0;\ \theta_1 = \theta_2 = \pi;\ \theta_1 = 0, \theta_2 = \pi;\ \theta_1 = \pi, \theta_2 = 0\}$. These are precisely the points of the poles of the two S^2's in the original $T^{1,1}$ solution which are the vertices of the convex polytope over which the T^3 is fibered and gives $T^{1,1}$.

The Ricci-scalar exhibits this singularity (in fact the other scalar invariants diverge in a similar manner) and is given by

$$
\begin{aligned}
R = \frac{12m^2}{\mathrm{U}}\Big(&\sin^4\theta_1\big(8\cos^4\theta_2 - 27\cos^2\theta_2\sin^2\theta_2 - 27\sin^4\theta_2\big) \\
&+ \cos^2\theta_1\sin^2\theta_1\big(36\cos^2\theta_2 + 28\cos^2\theta_2\sin^2\theta_2 - 27\sin^4\theta_2\big) \\
&+ 4\cos^4\theta_1\big(9\cos^2\theta_2\sin^2\theta_2 + 2\sin^4\theta_2\big)\Big) .
\end{aligned}
$$

5.4.3 Supersymmetry

To put the following solution into the above notation it is simplest to compute the Killing spinors in order to evaluate the spinor bilinears. Rather than computing the Killing spinors of the more difficult double T-dualized solution, the easier procedure is to compute the Killing spinors for $T^{1,1}$ and then map them under the two T-dualities. In the frame used to perform the first T-duality the Killing spinor is given by,

$$
\xi_1 \equiv \chi_1 + \mathrm{i}\chi_2 = c_1 e^{\mathrm{i}\frac{\psi}{2} + \frac{\mathrm{i}}{2}\arctan\left(\sqrt{\frac{3}{2}}\tan\theta_1\right)} \begin{pmatrix} 0 \\ -\frac{\sqrt{\sqrt{6}\cos\theta_1 - 3\mathrm{i}\sin\theta_1}}{\sqrt{\sqrt{6}\cos\theta_1 + 3\mathrm{i}\sin\theta_1}} \\ 0 \\ 1 \end{pmatrix} . \qquad (5.54)
$$

As the solution is Sasaki-Einstein the second Killing spinor $\xi_2 \equiv \chi_1 - \mathrm{i}\chi_2 = 0$ and fixes this $\chi_1 = \mathrm{i}\chi_2$.

Under a T-duality the Killing spinors transforms as

$$\epsilon_1 \to \epsilon_1 \ , \qquad \epsilon_2 \to \Omega_{U(1)}\epsilon_2 \tag{5.55}$$

with

$$\Omega_{U(1)} = \frac{1}{\sqrt{G_{99}}}\Gamma^{11}\Gamma_9 \tag{5.56}$$

where the index 9 is curved and is the direction along which the T-duality is performed. Into the notation of the 5d Killing spinors these transformations become

$$\chi_1 \to \chi_1 \ , \qquad \chi_2 \to \frac{1}{\sqrt{G_{99}}}\gamma_9\chi_2 \ , \tag{5.57}$$

with γ_9 the d=5 gamma matrix with curved index along the Killing direction.

The explicit Killing spinor is not very enlightening and therefore we do not present it here explicitly but just the results. We are now able to put the solution into the notation of the classification. We find for the scalars

$$\begin{aligned} A &= 1 \quad \Leftrightarrow \quad c_1 = \frac{1}{\sqrt{2}} \ , \\ \sin\zeta &= 0 \ , \\ Z &= 0 \\ S &= -\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\psi}}{\sqrt{U}} \ . \end{aligned} \tag{5.58}$$

From these we may extract out that the following dictionary

$$\begin{aligned} \psi_c &= -\psi \\ \mu &= \mathrm{e}^{\Phi_{IIB}} = \frac{1}{\lambda_1\sqrt{U}} \\ \eta &= 3\lambda\lambda_1^2\sin\theta_1\sin\theta_2 \end{aligned} \tag{5.59}$$

where ψ_c is the Killing direction of the classification. Computing the one forms we find

$$\begin{aligned} \beta &= \cos\theta_2\mathrm{d}\varphi_1 - \cos\theta_1\mathrm{d}\varphi_2 \\ \mathrm{Re}\,[\sigma] &= -\frac{\lambda_1^{3/2}}{U^{1/4}}(\cos\theta_1\sin^2\theta_2\mathrm{d}\varphi_1 + \cos\theta_2\sin^2\theta_1\mathrm{d}\varphi_2) \\ \mathrm{Im}\,[\sigma] &= \lambda_1^{5/2}U^{1/4}(\cos\theta_1\sin\theta_2\mathrm{d}\theta_2 - \cos\theta_2\sin\theta_1\mathrm{d}\theta_1) \ . \end{aligned} \tag{5.60}$$

Notice that with these identifications the fluxes are given exactly by the form presented in [122].

5.4.4 Central charge computation

As the solutions are T-dual the field central charges of both solutions should be identical. This is slightly complicated by the fact that the T-dualized solutions are singular. However the relevant formulae for computing the central chargeis not singular and the naive computation is correct. For the five-dimensional Newton's constant we shall use the formula given in appendix E of [7] for a warped AdS_5 metric in Einstein frame of the form

$$\mathrm{d}s^2 = \mathrm{e}^{2H}(\mathrm{d}s^2(\mathrm{AdS}_5) + \mathrm{d}s^2(\mathcal{M}_5)) \ . \tag{5.61}$$

The relevant formula reads

$$\frac{1}{16\pi G_5} = \frac{1}{2\kappa_{10}^2}\int_{M_5} \mathrm{e}^{8H}\mathrm{dvol}(\mathcal{M}_5) \ . \tag{5.62}$$

For our solution we identify the warp-factor to be $\mathrm{e}^{4H} = \mathrm{e}^{-\Phi}$ which arises from transforming from string-frame to Einstein-frame. Computing the integrand for the double T-dual solution we have

$$\frac{1}{16\pi G_5} = \frac{1}{2\kappa_{10}^2}\int_{M_5} \mathrm{dvol}(T^{1,1}) \ , \tag{5.63}$$

where $\mathrm{dvol}(T^{1,1})$ is the volume form on $T^{1,1}$. As the periods of the two φ coordinates remain the same under the T-dualities, it is obvious that the central charges of the two solutions will be the same[6].

We must check that the fluxes are properly quantised. We shall first perform the quantisation of the original solution and see how this maps into the double T-dual solution.

To quantise the five-form flux we impose that through all cycles $C_a \in H_5(\mathcal{M}_5, \mathbb{Z})$

$$N = \frac{1}{(2\pi\ell_{\mathrm{s}})^4 g_{\mathrm{s}}}\int_{\mathcal{M}_5} \mathrm{d}C^{(4)} \in \mathbb{Z} \ . \tag{5.64}$$

It is clear that there is only one five-cycle to integrate over, the compact internal manifold itself, and the condition becomes

$$N = \frac{1}{(2\pi\ell_{\mathrm{s}})^4 g_{\mathrm{s}}}\int_{T^{1,1}} \frac{4}{m^4}\mathrm{dvol}(T^{1,1}) \tag{5.65}$$

[6]Note that we have used the Klebanov Witten solution with constant overall warp-factor being 1. One may add an arbitrary constant however this may be absorbed into the definition of the inverse radius of AdS_5, m and therefore without loss of generality we do not add such a term. Furthermore we have taken the $\Phi = 0$, again one may put this to be an arbitrary constant, however as before this may be absorbed by a redefinition.

which defines the integer

$$N = \frac{4\mathrm{vol}(T^{1,1})}{(2\pi\ell_{\mathrm{s}}m)^4 g_{\mathrm{s}}} \ . \tag{5.66}$$

Now we turn to the quantisation of the flux in the double T-dual solution consisting of the RR three-form flux and the field strength of the NS-NS two-form through all three-cylces $C_a^{(3)} \in H_3(\mathcal{M}_5, \mathbb{Z})$. For the RR three-form flux the quantisation condition is

$$N(C_a^{(3)}) = \frac{1}{(2\pi\ell_{\mathrm{s}})^2 g_{\mathrm{s}}} \int_{C_a^{(3)}} \mathrm{d}C^{(2)} \in \mathbb{Z} \ . \tag{5.67}$$

For our purposes there is only one relevant three-cycle which is the one at fixed φ_i. We obtain

$$\widetilde{N} = \frac{4\mathrm{vol}(T^{1,1})}{(2\pi\ell_{\mathrm{s}}m)^2 \tilde{g}_{\mathrm{s}}(2\pi)^2} \ . \tag{5.68}$$

In fact this is precisely the same N as appeared in the original solution. To see this we recall that the string coupling constant after the two T-dualities becomes $\tilde{g}_{\mathrm{s}} = m^2 {\ell_{\mathrm{s}}}^2 g_{\mathrm{s}}$. Using this identity we find $N = \widetilde{N}$. This was to be expected on physical grounds, as the D5's are the result of the two T-dualities performed along the coordinates transverse to the N D3-branes.

Of course the choice of toric Sasaki-Einstein manifold was immaterial, any toric Sasaki-Einstein manifold would work just as well. In hindsight it is obvious that the above procedure may be interpreted as the infinite limit of a beta deformation a'la [121], where in this limit the D3-branes have now disappeared. Though the solutions are singular note that we have been able to obtain a well-defined central charge. This is to be expected as under a T-duality the Einstein frame volume measure is invariant and also agrees with our expectation following from the marginal deformation that is a beta deformation. It would be interesting to see if one can identify the singularities as the result of a D5-brane web as one would expect such a singularity for such a source. If such a setup exists, in principle one should be able to understand the dual field theories and to compute the central charge directly from the field theory to match with the above like in the usual Sasaki-Einstein case.

5.5 The solution of [1]

Part of the motivation for completing this work was to clarify the geometry underlying the two supersymmetric solutions in [1] which circumvented the classification of [7]. In this final section we show that the supersymmetric NATD-T dual of the AdS$_5\times$ T$^{(1,1)}$ solution in [1] satisfies our classification. We were unable to directly solve the equations of the classification to recover the solution (due to the complexity of the equations), as was done in [7] for the Pilch-Warner solution. We instead bypassed this problem by finding the Killing spinors from which we constructed the geometry by way of the spinor bilinears. We first begin this section by writing down the solution found in [1].

We use the coordinates $x_1 = \rho \sin\chi$, $x_2 = \rho\cos\chi$ and for simplicity set $\alpha' = 1$. The d=10 metric in string frame[7] is

$$\begin{aligned}\mathrm{d}s^2 = \mathrm{d}s^2(AdS_5) + L^2\lambda_1^2\mathrm{d}\theta_1^2 + \frac{1}{L^2P\mathcal{Q}}\big((L^4\lambda^2\lambda_1^2 + x_1^2)\mathrm{d}x_1 + x_1x_2\mathrm{d}x_2\big)^2 \\ +\frac{L^2\lambda_1^2}{P}\mathrm{d}x_2^2 + \frac{1}{L^2W\mathcal{Q}}(\mathcal{Q}\mathrm{d}\phi_1 - \lambda^2x_1x_2\cos\theta_1\mathrm{d}x_1 - \lambda^2(L^4\lambda_1^4 + x_2^2)\cos\theta_1\mathrm{d}x_2)^2 \\ +\frac{L^2\lambda^2\lambda_1^4x_1^2\sin^2\theta_1}{W}\mathrm{d}\xi^2 \ ,\end{aligned} \tag{5.69}$$

where

$$\mathcal{Q} = L^4\lambda^2\lambda_1^4 + \lambda_1^2x_1^2 + \lambda^2x_2^2 \ , \quad W = \lambda_1^2\mathcal{Q}\sin^2\theta_1 + \lambda^2\lambda_1^2x_1^2\cos^2\theta_1 \ , \quad P = L^4\lambda^2\lambda_1^2 + x_1^2 \ .$$

The constants λ and λ_1 take the values $1/3$ and $1/\sqrt{6}$ respectively and L is the radius of AdS_5. The dilaton is

$$\mathrm{e}^{-2\Phi} = L^4W \ , \tag{5.70}$$

whilst the NS-NS two-form is given by[8]

$$B = -\frac{\lambda_1^2x_1}{W}\bigg(\lambda^2x_1\cos\theta_1\mathrm{d}\phi_1 + \lambda^2x_2\sin^2\theta_1\mathrm{d}x_1 - x_1(\lambda^2\cos^2\theta_1 + \lambda_1^2\sin^2\theta_1)\mathrm{d}x_2\bigg)\wedge\mathrm{d}\xi \ . \tag{5.71}$$

The non-zero RR-fluxes[9] are

$$F_1 = 4L^4\lambda\lambda_1^4\sin\theta_1\mathrm{d}\theta_1 \ , \tag{5.72}$$

$$\begin{aligned}F_3 = \frac{4L^4\lambda\lambda_1^6x_1\sin\theta_1}{W}\mathrm{d}\theta_1\wedge\mathrm{d}\xi\wedge \\ \big[\lambda^2x_2\sin^2\theta_1\mathrm{d}x_1 - x_1(\lambda^2\cos^2\theta_1 + \lambda_1^2\sin^2\theta_1)\mathrm{d}x_2 + \lambda^2x_1\cos\theta_1\mathrm{d}\phi_1\big]\end{aligned} \tag{5.73}$$

and of course their hodge duals. In the notation of this classification the correspond-

[7]Recall that the classification is in Einstein frame.

[8]We correct a minor typographical error here by adding the $\cos\theta_1$ term in front of $\mathrm{d}\phi_1$.

[9]These are the ones that appear in the equations of motion, $F_n = \mathrm{d}C_{n-1} - C_{n-3}\wedge\mathrm{d}B$.

ing elements are

$$\begin{aligned}
m &= \frac{1}{L} \ , && (5.74)\\
\eta &= L^2\lambda_1^2 x_1 \sin\theta_1 \ , && (5.75)\\
\mu &= \frac{1}{L^2\sqrt{W}} = \mathrm{e}^{\Phi} \ , && (5.76)\\
\mathrm{d}\psi &= -\mathrm{d}\xi \ , && (5.77)\\
\beta &= (-x_1\cos\theta_1\mathrm{d}x_1 - x_2\cos\theta_1\mathrm{d}x_2 + L^4\lambda_1^4\sin\theta_1\mathrm{d}\theta_1 + x_2\mathrm{d}\phi_1) \ , && (5.78)\\
\sigma &= \frac{L\lambda_1^2}{W^{1/4}}[x_1x_2\sin^2\theta_1\mathrm{d}x_1 + (L^4\lambda_1^4 + x_2^2)\sin^2\theta_1\mathrm{d}x_2 + x_1^2\cos\theta_1\mathrm{d}\phi_1 \\
&\quad +\mathrm{i}L^2\sqrt{W}(\cos\theta_1\mathrm{d}x_2 + x_2\sin\theta_1\mathrm{d}\theta_1 - \mathrm{d}\phi_1)] \ . && (5.79)
\end{aligned}$$

Further details on the derivation of this dictionary is presented in appendix D.2. One may check that (5.69) takes the form of (5.19) with these identifications. For the explicit form of the NS-NS two form we find

$$B = \frac{\mathrm{e}^{\Phi/2}\mu}{9m^2}\mathrm{Re}\,[\sigma]\wedge\mathrm{d}\psi - \mathrm{d}x_2\wedge\mathrm{d}\psi \ , \tag{5.80}$$

whilst $C^{(2)}$ is not given in [1] for us to compare with, however it is trivial to show that F_3 agrees with that derived from the general expressions (5.44) and (5.45).

We have checked that this solution satisfies all the conditions of the classification, as an illustrative example we present the solution of (5.33). First define the function $E = (L^4\lambda_1^4 + x_2^2)\sin^2\theta_1 + x_1^2\cos^2\theta_1$. A short calculation gives

$$\mathrm{i}\sigma^*\wedge\sigma - 2\eta\mathrm{d}\eta\wedge\beta = 2L^{16}\lambda_1^4 E[\mathrm{d}x_2\wedge\mathrm{d}\phi_1 + x_1\sin\theta_1\mathrm{d}\theta_1\wedge\mathrm{d}x_1 + x_2\sin\theta_1\mathrm{d}\theta_1\wedge\mathrm{d}x_2] \ , \tag{5.81}$$

whilst

$$\frac{3(1-\eta^2\mu^2)}{\mu^2}\mathrm{d}\beta = L^{16}\lambda\lambda_1^2 E[\mathrm{d}x_2\wedge\mathrm{d}\phi_1 + x_1\sin\theta_1\mathrm{d}\theta_1\wedge\mathrm{d}x_1 + x_2\sin\theta_1\mathrm{d}\theta_1\wedge\mathrm{d}x_2] \ . \tag{5.82}$$

Upon substituting the values of the constants, λ and λ_1 we find that they are equal. The equation for σ follows similarly but is vastly more complicated than the one illustrated above and for this reason we do not present it.

In section 5.3 we saw that the integrable almost product structure implied that the one-forms β and σ had no $\mathrm{d}\tau$ term, we would like to verify this. To do so we must write the one-forms in the form (5.42). To this end, we make the change of coordinates

$$
\begin{aligned}
x &= \phi_1 \ , && (5.83)\\
y_1 &= \frac{1}{2}(x_1^2 + x_2^2) + L^4\lambda_1^4 \ln(\cos\theta_1) \ , && (5.84)\\
y_2 &= \ln\left(\frac{x_2}{\cos\theta_1}\right) \ , && (5.85)\\
\eta &= L^2\lambda_1^2 x_1 \sin\theta_1 \ . && (5.86)
\end{aligned}
$$

In these coordinates the coefficients for the one-forms, in the notation of (5.42), are

$$
\begin{aligned}
\gamma_x &= x_2 \ , & \gamma_{y_1} &= -\cos\theta_1 \ , & \gamma_{y_2} &= 0 \ , && (5.87)\\
\rho_x &= \frac{L\lambda_1^2 x_1^2\cos\theta_1}{W^{1/4}} \ , & \rho_{y_1} &= \frac{L\lambda_1^2 x_2 \sin^2\theta_1}{W^{1/4}} \ , & \rho_{y_2} &= \frac{L^5\lambda_1^6 x_2\sin^2\theta_1}{W^{1/4}} \ , && (5.88)\\
\kappa_x &= -L^3\lambda_1^2 W^{1/4} \ , & \kappa_{y_1} &= 0 \ , & \kappa_{y_2} &= L^3\lambda_1^2 x_2 \cos\theta_1 W^{1/4} && (5.89)
\end{aligned}
$$

It is clear that this satisfies the integrable almost product structure. We have again checked that with these new coordinates the equations of the classification are satisfied and once again the equations to solve are very complicated. We had hoped this solution would have motivated further ansatz, unfortunately this was not the case. Interestingly this solution has an additional Killing vector, ∂_x, to what the classification implies. Imposing this extra Killing direction does not give much in the way of simplification of the equations and so this ansatz was swiftly dropped in favour of the ones we have presented.

We note that this solution, like our one, is singular [1]. The Ricci tensor blows up as $\theta_1 \to 0$ or π whilst $x_1 \to 0$. Furthermore the dilaton also blows up at these points. Computing the invariants $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu_1..\mu_4}R^{\mu_1..\mu_4}$ we also find that these are singular at these points but only these points. This solution therefore exhibits two singular points.

Though the solution is singular it would still be interesting to interpret this solution's field theory dual and also its brane realisation. A method was proposed in [127] where they considered the type IIA non-Abelian T dual of $\mathrm{AdS}_5 \times S^5$ and propose a a D4/NS5 brane set-up and a linear quiver to describe its dual SCFT.

In [1] they also present another supersymmetric Type IIB solution with $F_5 = 0$, namely the NATD-T dual of the $\mathrm{AdS}_5 \times Y^{p,q}$ solution. This solution will also satisfy the classification presented here however we do not present the details.

5.6 Complex $\mathcal{M}_4$ and $P = 0$

Motivated by finding explicit solutions we set $P = 0$ in this section[10]. Notice that setting $P = 0$ implies that μ is a function of η only[11]. Setting $P = 0$ and $\mu = \mu(\eta)$ reduces the necessary and sufficient differential equations to

$$
\begin{aligned}
\mathrm{d}_3\beta &= \frac{2\mu^2}{3(1-\eta^2\mu^2)}\mathrm{Im}\,[\sigma] \wedge \mathrm{Re}\,[\sigma] \ , && (5.92)\\
\mathrm{d}_3\mathrm{Re}\,[\sigma] &= \frac{\mu\eta}{\eta^2\mu^2-1}\partial_\eta \ln\mu \ \beta \wedge \mathrm{Im}\,[\sigma] \ , && (5.93)\\
\mathrm{d}_3\mathrm{Im}\,[\sigma] &= -\frac{\mu\eta}{\eta^2\mu^2-1}\partial_\eta \ln\mu \ \beta \wedge \mathrm{Re}\,[\sigma] \ , && (5.94)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_\eta\beta &= -\frac{2\eta\mu^2}{3(1-\eta^2\mu^2)}\beta \ , && (5.95)\\
\partial_\eta\mathrm{Re}\,[\sigma] &= \frac{1}{\eta^2\mu^2-1}\left(\frac{4\mu^2\eta}{3} + \partial_\eta \ln\mu\right)\mathrm{Re}\,[\sigma] \ , && (5.96)\\
\partial_\eta\mathrm{Im}\,[\sigma] &= \frac{1}{\eta^2\mu^2-1}\left(\frac{4\mu^2\eta}{3} + \partial_\eta \ln\mu\right)\mathrm{Im}\,[\sigma] \ . && (5.97)
\end{aligned}
$$

We see immediately that we may solve (5.95) - (5.97) as

$$
\begin{aligned}
\beta &= \exp\left[\int \frac{2\mu^2\eta}{3(\eta^2\mu^2-1)}\mathrm{d}\eta\right]\hat{\beta} \ , && (5.98)\\
\mathrm{Re}\,[\sigma] &= \exp\left[\int \frac{1}{\eta^2\mu^2-1}\left(\frac{4\mu^2\eta}{3} + \partial_\eta \ln\mu\right)\mathrm{d}\eta\right]\hat{R} \ , && (5.99)\\
\mathrm{Im}\,[\sigma] &= \exp\left[\int \frac{1}{\eta^2\mu^2-1}\left(\frac{4\mu^2\eta}{3} + \partial_\eta \ln\mu\right)\mathrm{d}\eta\right]\hat{I} \ , && (5.100)
\end{aligned}
$$

---

[10] This condition imposes that the distinguished transverse four-dimensional foliation defined by the Killing vector ∂_ψ, which we call M_4, has an integrable almost complex structure. Consider a holomorphic two-form constructed from the orthonormal frame of appendix D.1 as

$$
\begin{aligned}
\Omega &\equiv (e^2 + \mathrm{i}e^5) \wedge (e^4 - \mathrm{i}e^3)\\
&= \frac{1}{2(\eta\mu-1)}(\mathrm{e}^{\mathrm{i}\psi}X + \mathrm{e}^{-\mathrm{i}\psi}Y^* + 2W) \ . && (5.90)
\end{aligned}
$$

This then defines an almost complex structure on M_4. In the second line we have expressed Ω in terms of the two-form bilinears. Imposing that this is integrable implies

$$
P = g(e^4 + \mathrm{i}e^3) + f(e^2 + \mathrm{i}e^5) + h(e^4 - \mathrm{i}e^3) \ , \tag{5.91}
$$

where f, g, h are arbitrary complex functions (subject to satisfying the P equation of motion and Bianchi identity). Setting $P = 0$ solves this constraint therefore $\mathcal{M}_4$ is complex in this case. It would have been more interesting to impose this more general form of P, however it is still a fairly complicated system of equations to solve and we were unable to do so.

[11] To see this use (5.35) to note that $\mathrm{d}\ln\mu = f_{K_4}K_4 + f_\eta \mathrm{d}\eta$ for some real functions f_{K_4} and f_η. Requiring that this is closed then implies that $f_{K_4} = 0$.

where the hatted objects are η independent one-forms. We note that the above integrations may include arbitrary integration constants which we absorb into the η independent one-forms. Upon substituting these expressions into (5.92) - (5.94) one sees that the η dependence in (5.92) cancels automatically as it should. However the η dependence in (5.93) and (5.94) does not, we should have been suspicious if it cancelled as it would imply that μ could be any function of η, requiring that this expression is η independent gives us the defining differential equation for μ

$$\partial_\eta\left(\frac{\mu\eta}{\eta^2\mu^2-1}\partial_\eta \ln\mu \exp\left[\int\frac{2\mu^2\eta}{3(\eta^2\mu^2-1)}\mathrm{d}\eta\right]\right)=0\ . \tag{5.101}$$

We find a solution to the system of differential equations if we satisfy the second order non-linear differential equation

$$(3+\eta^2\mu^2)\dot{\mu}+6\eta^3\mu\dot{\mu}^2+3\eta(1-\eta^2\mu^2)\ddot{\mu}=0\ , \tag{5.102}$$

and the three differential equations

$$\mathrm{d}_3\hat{\beta} = \frac{2}{3}\hat{I}\wedge\hat{R}\ , \tag{5.103}$$

$$\mathrm{d}_3\hat{R} = c\hat{\beta}\wedge\hat{I}\ , \tag{5.104}$$

$$\mathrm{d}_3\hat{I} = -c\hat{\beta}\wedge\hat{R}\ . \tag{5.105}$$

Where c is a constant satisfying

$$c=\frac{\mu\eta}{\eta^2\mu^2-1}\partial_\eta \ln\mu \exp\left[\int\frac{2\mu^2\eta}{3(\eta^2\mu^2-1)}\mathrm{d}\eta\right]\ . \tag{5.106}$$

Notice that c is non-zero if μ is non-constant and we shall distinguish between these two cases. For the $c=0$ case we can write the solution in closed form and we will discuss it in the remainder of this section. However we are unable to write the $c\neq 0$ case in closed form.

A singular solution

We look at the $c=0$ solution of (5.102) which is equivalent to constant μ. For simplicity we set $\mu=1$. We are now able to integrate (5.98)- (5.100); we find

$$\beta = (1-\eta^2)^{1/3}\hat{\beta}\ , \qquad \partial_\eta\hat{\beta}=0\ , \tag{5.107}$$

$$\mathrm{Re}\,[\sigma] = (1-\eta^2)^{2/3}\hat{R}\ , \qquad \partial_\eta\hat{R}=0\ , \tag{5.108}$$

$$\mathrm{Im}\,[\sigma] = (1-\eta^2)^{2/3}\hat{I}\ , \qquad \partial_\eta\hat{I}=0\ . \tag{5.109}$$

We then need to solve

$$\mathrm{d}_3\hat{R} = 0 = \mathrm{d}_3\hat{I} \ , \tag{5.110}$$

$$\mathrm{d}_3\hat{\beta} = \frac{2}{3}\hat{I}\wedge\hat{R} \ . \tag{5.111}$$

As $\hat{R}$ and $\hat{I}$ are closed we may define coordinates y_1 and y_2 such that

$$\hat{R} = \mathrm{d}y_2 \ , \quad \hat{I} = \mathrm{d}y_1 \ . \tag{5.112}$$

A solution to (5.111) is

$$\hat{\beta} = \frac{2}{3}(\mathrm{d}x + y_1\mathrm{d}y_2) \ . \tag{5.113}$$

The metric is

$$\begin{aligned} 9m^2\mathrm{d}s^2(\mathcal{M}_5) &= \eta^2\mathrm{d}\psi^2 + (1-\eta^2)^{1/3}(\mathrm{d}y_1^2+\mathrm{d}y_2^2) \\ &\quad + \frac{4}{9(1-\eta^2)^{1/3}}(\mathrm{d}x+y_1\mathrm{d}y_2)^2 + \frac{1}{1-\eta^2}\mathrm{d}\eta^2 \ , \end{aligned} \tag{5.114}$$

and we have

$$B = \frac{(1-\eta^2)^{2/3}}{9m^2}\mathrm{d}y_2\wedge\mathrm{d}\psi \ , \tag{5.115}$$

$$C^{(2)} = \frac{(1-\eta^2)^{2/3}}{9m^2}\mathrm{d}y_1\wedge\mathrm{d}\psi \ . \tag{5.116}$$

Note that the range of η should be either $\eta\in[0,1]$ or $\eta\in[-1,0]$. We find that the Ricci Scalar is given by $R = 28m^2$, whilst $R_{\mu\nu}R^{\mu\nu} = 336m^4$ however we find that $R_{\mu_1..\mu_4}R^{\mu_1..\mu_4}$ exhibits a singularity as $\eta\to\pm1$ and therefore the solution is singular. We note that for $F_5\neq0$ an analogous solution of the equations of [7] exists, which was missed previously, by setting $\phi, C^{(0)}$ and the warp factor to be constants. This solution is once again singular and the singularity appears first in the Ricci scalar, it has non-zero G and hence is also not Sasaki-Einstein. These solutions are unusual in the sense that the only other known solutions with constant warp factor are the Sasaki-Einstein solutions.

5.7 Concluding remarks

This work has plugged the remaining gap in the classification of all AdS_5 supersymmetric solutions of Type IIB supergravity. Together with [7, 31, 123] our work concludes the classification of all supersymmetric AdS_5 solutions of $d = 10$ and $d = 11$ supergravity. We find that the geometry of M_5 is different to that of the $F_5\neq0$ case. It should be possible to interpret these results in terms of the "Exceptional Sasaki-Einstein (ESE) geometry" of [128]. It would be interesting to see how the ESE structure is interpreted in terms of the bilinears. A similar analysis was carried out in [128] for the case of $F_5 \neq 0$.

Chapter 6

Conclusions and future directions

In this thesis we have argued for the marriage of F-theory and holography. We have shown that we are able to make non-trivial predictions that are otherwise difficult to prove without appealing to this marriage. One may extend this to different dimensions in particular understanding the 4d theories dual to the solutions discussed in section 2.4.3. Extensions to the holographic dictionary in F-theory to higher dimensions could build also on the work in [42–44], for AdS_6 F-theoretic solutions, which could be put into a more F-theoretic friendly setting.

Work is being undertaken to extend the above classification of AdS_3 solutions to include general fluxes and thus extend the work of [56] in including three-form fluxes. It is easy to see that the current literature is incomplete. For example the classic D1-D5 brane intersection with metric $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ does not fall into any existing classification despite the rich literature in identifying the field theory dual. Understanding the underlying geometry of such solutions not only allows for a more systematic search for such solutions but may also allow for a better understanding of the dual field theories.

During the course of this thesis we have noted some problems with the matching of c-extremisation and gravity in [58] in the constant axio-dilaton case. This is a surprising result and deserves further investigation. Preliminary results indicate that this mismatch of the various ranges of the defining integers, $\mathfrak{p}, \mathfrak{q}$ here, also lifts to more general field theories compactified on T^2 with Baryonic twist. The field theories dual to the $Y^{p,q}$ Sasaki-Einstein solutions, can be obtained by blowing down the $X^{p,q}$ quiver theories [129] which in turn may be seen as the blow down of the $Z^{p,q}$ quiver field theories [130]. It is curious that the same result extends, though not unexpected given the relation between the various field theories, to these other field theories. It seems to indicate that such a 2d IR fixed point does not exist however this raises two issues; what is the fate of the field theory on T^2 and what is the supergravity solution dual to? Work is being undertaken in this direction by the author and collaborators.

The geometries in section 5 also deserve further exploration. There are very few known solutions and all have metric singularities. It is an open problem as to whether all these solutions are necessarily singular, of course, as we have tried to explain during this thesis, singular metrics are not necessarily bad when the singularities have a physical interpretation. It would be natural to interpret these singularities as arising from 5-branes, and this is something that needs verification in the future. One could also extend the work following that of [124] by giving a general formula for computing the central charge and R-charges from general geometric data. The geometries also have the novel feature that the R-symmetry vector is unfibered, there are few examples of this phenomenon occurring and the fact that all geometries of this type are of this form leads this to be an interesting testing ground.

Appendix A

G-structure analysis for AdS$_3$ duals

A.1 Conventions for Gamma Matrices and Spinors

We shall use the letters $M, N, ..$ for the 10d indices, $a, b, ..$ takes values $0, 1, 2$ and are used for the AdS$_3$ indices and $\mu, \nu, .. \in \{1, .., 7\}$ for the indices for $\mathcal{M}_7$. Following [131] we decompose the 10d Gamma matrices as

$$\Gamma^a = \rho^a \otimes 1 \otimes \sigma_2 \,, \tag{A.1}$$

$$\Gamma^\mu = 1 \otimes \gamma^\mu \otimes \sigma_1 \,, \tag{A.2}$$

where ρ^a generate Cliff(1,2) and γ^μ generate Cliff(7). Explicitly we shall take

$$\rho_0 = \mathrm{i}\ \sigma^1 \,, \quad \rho_1 = \sigma^2 \,, \quad \rho_2 = \sigma^3 \,, \tag{A.3}$$

with $\rho_{012} = -1$. For the Cliff(7) gamma matrices we shall take

$$\gamma^1 = -\sigma_1 \otimes 1 \otimes 1 \,, \tag{A.4}$$

$$\gamma^2 = \sigma_3 \otimes \sigma_2 \otimes 1 \,, \tag{A.5}$$

$$\gamma^3 = -\sigma_2 \otimes 1 \otimes 1 \,, \tag{A.6}$$

$$\gamma^4 = \sigma_3 \otimes \sigma_1 \otimes 1 \,, \tag{A.7}$$

$$\gamma^5 = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \,, \tag{A.8}$$

$$\gamma^6 = -\sigma_3 \otimes \sigma_3 \otimes \sigma_2 \,, \tag{A.9}$$

$$\gamma^7 = -\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \,, \tag{A.10}$$

and we have $\gamma_{1...7} = -\mathrm{i}1$.With these conventions we have

$$\Gamma_{11} = \mathbb{1} \otimes \mathbb{1} \otimes \sigma_3 \,. \tag{A.11}$$

We follow the definitions in [132] for the various intertwiners. For the A intertwiner we have

$$\begin{aligned}
A_{10}\Gamma_M A_{10}^{-1} &= \Gamma_M^\dagger\,, && \text{(A.12)}\\
A_7\gamma_\mu A_7^{-1} &= \gamma_\mu^\dagger\,, && \text{(A.13)}\\
A_{1,2}\rho_a A_{1,2}^{-1} &= -\rho_a^\dagger\,, && \text{(A.14)}\\
A_{10} &= A_{1,2}\otimes A_7\otimes\sigma_1\,, && \text{(A.15)}\\
A_7 &= 1\,, && \text{(A.16)}\\
A_{1,2} &= \sigma_1\,. && \text{(A.17)}
\end{aligned}$$

For the charge conjugation intertwiner C we take

$$\begin{aligned}
C_{10}^{-1}\Gamma_M C_{10} &= -\Gamma_M^{\mathrm{T}}\,, && \text{(A.18)}\\
C_7^{-1}\gamma_\mu C_7 &= -\gamma_\mu^{\mathrm{T}}\,, && \text{(A.19)}\\
C_{1,2}^{-1}\rho_a C_{1,2} &= -\rho_a^{\mathrm{T}}\,, && \text{(A.20)}\\
C_{10} &= C_{1,2}\otimes C_7\otimes\sigma_1\,, && \text{(A.21)}\\
C_7 &= \sigma_2\otimes\sigma_1\otimes\sigma_2\,, && \text{(A.22)}\\
C_{1,2} &= \sigma_2\,. && \text{(A.23)}
\end{aligned}$$

We have

$$C_{10}^{\mathrm{T}} = -C_{10}\,,\quad C_7^{\mathrm{T}} = C_7\,,\quad C_{1,2}^{\mathrm{T}} = -C_{1,2}\,. \tag{A.24}$$

Finally the D intertwiner satisfies

$$\begin{aligned}
D_{10}^{-1}\Gamma_M D_{10} &= \Gamma_M^*\,, && \text{(A.25)}\\
D_7^{-1}\gamma_\mu D_7 &= -\gamma_\mu^*\,, && \text{(A.26)}\\
D_{1,2}^{-1}\rho^a D_{1,2} &= \rho_a^*\,, && \text{(A.27)}\\
D_{10} &= D_{1,2}\otimes D_7\otimes\sigma_3\,, && \text{(A.28)}\\
D_7 &= \sigma_2\otimes\sigma_1\otimes\sigma_2\,, && \text{(A.29)}\\
D_{1,2} &= -\mathrm{i}\,\sigma_3 && \text{(A.30)}
\end{aligned}$$

They satisfy

$$D_{10}^* = D_{10}^{-1}\,,\qquad D_7^* = D_7^{-1}\,,\qquad D_{1,2}^* = D_{1,2}^{-1}\,. \tag{A.31}$$

We now wish to decompose a 10d Majorana-Weyl spinor consistent with these conventions. We shall decompose the spinor, ϵ as $\epsilon = \psi\otimes\chi\otimes\theta$ where ψ is a two-component spinor, χ an eight-component spinor and θ a two-component spinor. The chirality condition in 10d is

$$\Gamma_{11}\epsilon = -\epsilon \tag{A.32}$$

which is solved by

$$\sigma_3\theta = -\theta\,. \tag{A.33}$$

For the Majorana condition we impose that both χ and ψ are Majorana and also that θ is purely imaginary. Type IIB supersymmetry is parametrised by two 10d Majorana-Weyl spinors. We may complexify the two Majorana-Weyl spinors into

$$\epsilon = \psi_1 \otimes \xi_1 \otimes \theta \tag{A.34}$$

where $\xi = \chi_1 + \mathrm{i}\chi_2$ is a Dirac spinor. This will generically preserve $(0,2)$ supersymmetry however we are also interested in finding the equations for preserving $(0,4)$ explicitly and so the ansatz we use to accommodate both cases is

$$\epsilon = \psi_1 \otimes \mathrm{e}^{H/2}\xi_1 \otimes \theta + \psi_2 \otimes \mathrm{e}^{H/2}\xi_2 \otimes \theta\,. \tag{A.35}$$

The $(0,2)$ case is obtained by setting one of the ξ's to zero. The warp factor appears here for later convenience. Here the ψ_i are Killing spinors on AdS_3 and satisfy the most general Killing spinor equations for two Killing spinors on AdS_3

$$\nabla_a\psi_i = \frac{m}{2}\sum_{j=1}^{2} W_{ij}\rho_a\psi_j\,. \tag{A.36}$$

It is easily shown that W is a scalar and furthermore may be diagonalized with eigenvalues ±1, we shall therefore omit this term in the following.

A.1.1 Bilinear definitions

We begin by defining the bilinears that we shall employ in this thesis, further details may be found in [59]. We write the bilinears for a general number of independent spinors specifying to 1 or 2 in the main text.

- *Scalar Bilinears*

$$S_{ij} \equiv \bar{\xi}_i\xi_j\ , \tag{A.37}$$
$$A_{ij} \equiv \bar{\xi}^c_i\xi_j\ . \tag{A.38}$$

- *One-form Bilinears*

$$K^\mu_{ij} \equiv \bar{\xi}_i\gamma^\mu\xi_j\ , \tag{A.39}$$
$$B^\mu_{ij} \equiv \bar{\xi}^c_i\gamma^\mu\xi_j\ . \tag{A.40}$$

- *Two-form Bilinears*

$$U_{ij}^{\mu_1\mu_2} \equiv \bar{\xi}_i \gamma^{\mu_1\mu_2} \xi_j \ , \tag{A.41}$$
$$V_{ij}^{\mu_1\mu_2} \equiv \bar{\xi}_i^c \gamma^{\mu_1\mu_2} \xi_j \ . \tag{A.42}$$

- *Three-form Bilinears*

$$X_{ij}^{\mu_1\mu_2\mu_3} \xi_j \equiv \bar{\xi}_i \gamma^{\mu_1\mu_2\mu_3} \xi_j \ , \tag{A.43}$$
$$Y_{ij}^{\mu_1\mu_2\mu_3} \xi_j \equiv \bar{\xi}_i^c \gamma^{\mu_1\mu_2\mu_3} \xi_j \ . \tag{A.44}$$

Higher order bilinears are related to the above ones by Hodge duality.

A.1.2 Torsion conditions

This appendix summarises the torsion conditions relevant for section 2.2. They are the same as computed in [59, 60], and we refer the reader there for further details.

- *Scalar differential equations*

$$\mathrm{d}S_{ij} = \frac{\mathrm{i}m}{2}(\alpha_i - \alpha_j) K_{ij} \ , \tag{A.45}$$
$$\mathrm{e}^{-2H} \mathcal{D}(\mathrm{e}^{2H} A_{ij}) = -\frac{\mathrm{i}m}{2}(\alpha_i - \alpha_j) B_{ij} \ . \tag{A.46}$$

- *One-form differential equations*

$$\mathrm{e}^{-4H} \mathrm{d}\left(\mathrm{e}^{4H} K_{ij}\right) = -\mathrm{i}m(\alpha_i + \alpha_j) U_{ij} - S_{ij} \mathrm{e}^{-4H} F^{(2)} \tag{A.47}$$
$$\mathcal{D}(\mathrm{e}^{2H} B_{ij}) = 0 \tag{A.48}$$

- *Two-form differential equations*

$$\mathrm{e}^{-4H} \mathrm{d}(\mathrm{e}^{4H} U_{ij}) = -\frac{\mathrm{i}m}{2}(\alpha_i - \alpha_j) X_{ij} \ , \tag{A.49}$$
$$\mathrm{e}^{-6H} \mathcal{D}\left(\mathrm{e}^{6H} V_{ij}\right) = -\frac{3\mathrm{i}m}{2}(\alpha_i - \alpha_j) Y_{ij} + \mathrm{e}^{-4H} F^{(2)} \wedge B_{ij} \tag{A.50}$$

- *Three-form differential equations*

$$\mathrm{e}^{-8H} \mathrm{d}\left(\mathrm{e}^{8H} X_{ij}\right) = 2m(\alpha_i + \alpha_j) * X_{ij} - \mathrm{e}^{-4H} F^{(2)} \wedge U_{ij} \ , \tag{A.51}$$
$$\mathrm{e}^{-6H} \mathcal{D}\left(\mathrm{e}^{6H} Y_{ij}\right) = m(\alpha_i + \alpha_j) * Y_{ij} \tag{A.52}$$

- *Four-form differential equations*

$$\mathrm{e}^{-8H}\mathrm{d}\left(\mathrm{e}^{8H} * X_{ij}\right) = -\frac{3\mathrm{i}m}{2}(\alpha_i - \alpha_j) * U_{ij} \tag{A.53}$$

$$\mathrm{e}^{-10H}\mathcal{D}\left(\mathrm{e}^{10H} * Y_{ij}\right) = -\frac{5\mathrm{i}m}{2}(\alpha_i - \alpha_j) * V_{ij} - \mathrm{i}\mathrm{e}^{-4H}F^{(2)} \wedge Y_{ij} \ , \tag{A.54}$$

$$\mathrm{e}^{-6H}\mathcal{D}\left(\mathrm{e}^{6H} * Y_{ij}\right) = -\frac{\mathrm{i}m}{2}(\alpha_i - \alpha_j) * V_{ij} - \mathrm{e}^{-4H}A_{ij} * F^{(2)} \tag{A.55}$$

- *Five-form differential equations*

$$\mathrm{e}^{-8H}\mathrm{d}\left(\mathrm{e}^{8H} * U_{ij}\right) = \mathrm{i}m(\alpha_i + \alpha_j) * K_{ij} \ , \tag{A.56}$$

$$\mathrm{e}^{-10H}\mathcal{D}\left(\mathrm{e}^{10H} * V_{ij}\right) = \mathrm{i}m(\alpha_i + \alpha_j) * B_{ij} \tag{A.57}$$

- *Six-form differential equations*

$$\mathrm{e}^{-12H}\mathrm{d}\left(\mathrm{e}^{12H} * K_{ij}\right) = \mathrm{i}m(\alpha_i - \alpha_j)S_{ij}\mathrm{Vol}(\mathcal{M}_7) \ , \tag{A.58}$$

$$\mathrm{e}^{-10H}\mathcal{D}\left(\mathrm{e}^{10H} * B_{ij}\right) = -\frac{3\mathrm{i}m}{2}(\alpha_i - \alpha_j)A_{ij}\mathrm{Vol}(\mathcal{M}_7) \ . \tag{A.59}$$

Symmetrizing the covariant derivative of the vectors K_{ij}, we find

$$\nabla_{(\mu_1}K^{ij}_{\mu_2)} = \frac{\mathrm{i}m}{2}(\alpha_j - \alpha_i)g_{\mu_1\mu_2}S_{ij} \ , \tag{A.60}$$

which implies a Killing vector when the right-hand side vanishes.

There are also various algebraic conditions following from the supersymmetry conditions. First note, by using (2.8) one can derive

$$(\alpha_i + \alpha_j)A_{ij} = 0 \, . \tag{A.61}$$

Notice this condition implies that the scalars A_{11} and A_{22} must vanish irrespective of the value of α_i. Using (2.8) one can also derive

$$\partial_\mu H\bar{\xi}_i\gamma^\mu\xi_j \ = \ \frac{\mathrm{i}m}{2}(\alpha_j - \alpha_i)\bar{\xi}_i\xi_j \, . \tag{A.62}$$

Finally, we have two conditions involving the one-form P, which follow from (2.7)

$$P_\mu\bar{\xi}_i\gamma^\mu\xi_j \ = \ 0 \, , \tag{A.63}$$

$$P_\mu\bar{\xi}^c_i\gamma^\mu\xi_j \ = \ 0 \, . \tag{A.64}$$

A.1.3 Derivation of the "Master Equation"

In this appendix we provide an extensive discussion on the derivation of the "master equation" (2.36). Supersymmetry implies that a solution satisfies the Einstein equation and the Bianchi identity for $F^{(2)}$ but not the equation of motion for $F^{(2)}$. In this appendix we show that the equation of motion for $F^{(2)}$ is equivalent to (2.36). In [55] the $F^{(2)}$ equation of motion is shown to be equivalent to the differential equation

$$\Box_6 R - \frac{1}{2}R^2 + R_{\mu\nu}R^{\mu\nu} = 0 \tag{A.65}$$

on the Kähler base. We shall find that a similar equation governs the existence of a solution when τ becomes non-trivial.

In the main text it was shown that the internal space is a $U(1)$-fibration over a warped six-dimensional Kähler base. In the following it will be necessary to reduce along the Killing direction and to express everything in terms of the Kähler metric rather than the warped one, as such it is necessary to first clarify the notation we shall be using. We denote by $*_7$ the Hodge dual operator on the internal space, $*_6$ is the Hodge dual operator on the base of the $U(1)$ fibration and $\hat{*}_6$ the Hodge dual operator on the Kähler metric. The Ricci tensor, Ricci scalar and Ricci-form appearing are that of the Kähler metric and the Kähler two form is denoted by J.

Supersymmetry implies that the flux satisfies

$$\begin{aligned} m *_7 F^{(2)} &= *_7 \left(-2J - 4m\mathrm{e}^{4H}\mathrm{d}H \wedge K - \frac{1}{2}\mathrm{e}^{4H}\mathrm{d}\rho \right) \\ &= \frac{\mathrm{e}^{-4H}}{m^2} K \wedge J \wedge J - \frac{1}{m^3}\hat{*}_6 \mathrm{d}\mathrm{e}^{-4H} - \frac{1}{2m^2}\hat{*}_6(\mathfrak{R} + \mathrm{d}Q) \wedge K \ . \end{aligned} \tag{A.66}$$

Making use of the identities (which are easily proven)

$$\hat{*}_6 \mathfrak{R} = \frac{R}{4} J \wedge J - \mathfrak{R} \wedge J \ , \tag{A.67}$$

$$\hat{*}_6 P \wedge P^* = -\frac{\mathrm{i}|P|^2}{2} J \wedge J - P \wedge P^* \wedge J \tag{A.68}$$

we have

$$m *_7 F^{(2)} = -\frac{1}{8m^3}\hat{*}_6 \mathrm{d}(R - 2|P|^2) + \frac{1}{2m^2}\left(\mathfrak{R} \wedge J - \mathrm{i}P \wedge P^* \wedge J\right) \wedge K \tag{A.69}$$

Imposing (2.4) is then equivalent to

$$0 = \mathrm{d}\hat{*}_6 \mathrm{d}(R - 2|P|^2) + 2\mathfrak{R} \wedge \mathfrak{R} \wedge J + 4\mathrm{i}\mathfrak{R} \wedge P \wedge P^* \wedge J \ . \tag{A.70}$$

Taking the Hodge dual of the above and using the identities

$$\hat{*}_6 \mathfrak{R} \wedge \mathfrak{R} \wedge J = \frac{1}{4}R^2 - \frac{1}{2}R_{\mu\nu}R^{\mu\nu} \ , \tag{A.71}$$

$$\hat{*}_6 \mathfrak{R} \wedge P \wedge P^* \wedge J = -\mathrm{i}\left(\frac{1}{2}R|P|^2 - R_{\mu\nu}P^{\mu}P^{*\nu}\right) \tag{A.72}$$

one obtains

$$\hat{\Box}_6(R-2|P|^2)=\frac{1}{2}R^2-R_{\mu\nu}R^{\mu\nu}-2|P|^2R+4R_{\mu\nu}P^{\mu}P^{*\nu}\ , \tag{A.73}$$

where

$$\hat{\Box}_6=\hat{*}_6\mathrm{d}\hat{*}_6\mathrm{d}\ . \tag{A.74}$$

Equation (2.36) determines the Kähler metric from which the remaining geometry may be recovered. Notice that for constant axio-dilaton one recovers the equation of [55] as expected.

A.2 AdS$_3$ to AdS$_5$

In this appendix we provide some of the computational derivations for section 2.4.3. We look at the AdS$_3$ solutions with $\mathcal{N}=(2,2)$ and varying τ by relaxing the compactness condition of the internal space. We find the only solutions of this problem decompactify to an AdS$_5$ solution. In fact the resulting AdS$_5$ varying τ solutions of IIB supergravity are the most general of this kind. In [133] AdS$_5$ solutions with five-form flux and varying axio-dilaton were considered. We recover the analysis presented there and give an F-theoretic interpretation in terms of an elliptically fibered Calabi–Yau four-fold.

A.2.1 AdS$_3$ Solutions with $(2,2)$ and Varying τ

Torsion Conditions

The starting point for this analysis is (2.118) where for P to be non-zero and thus τ varying, we need $A_{12}=0$. It is easy to see that by setting S_{12} to be constant it must in fact vanish. Moreover it is trivial to see that it is impossible to satisfy the torsion conditions if both of these scalars simultaneously vanish. We shall therefore restrict to the case when S_{12} is non-constant in the remainder of this subsection. As before we find that both S_{11} and S_{22} are constant and therefore we may normalise the spinors such that they are both unity.

Recall that $\mathcal{M}_7$ admits an $SU(2)$ structure which implies there is a $3+4$ splitting, such that the '3' part, $\widetilde{M}_3$ has a vielbein given by the three vectors of the SU(2) structure. One may take as a basis for the three independent vectors

$$\{K_{11},\ K_{22},\ \mathrm{Im}\,[S_{12}^{*}K_{12}]\} \tag{A.75}$$

in terms of which we may write the metric on $\mathcal{M}_3$ as

$$\mathrm{d}s^2(\mathcal{M}_3) = \frac{1}{4|S_{12}|^2(1-|S_{12}|^2)}\left(K_{11}^2 + K_{22}^2 + 2(1-2|S_{12}|^2)K_{11}\otimes K_{22} + 4\mathrm{Im}\,[S_{12}^* K_{12}]^2\right) \, . \tag{A.76}$$

The canonical $SU(2)$ structure two-forms are written in terms of the bilinears as

$$j = \mathrm{i}U_{11} - \frac{1}{2|S_{12}|^2(1-|S_{12}|^2)}\left(K_{22} + (1-2|S_{12}|^2 K_{11}\right)\wedge \mathrm{Im}\,[S_{12}^* K_{12}] \, , \tag{A.77}$$

$$\omega = \frac{1}{(1-|S_{12}|^2)^{\frac{1}{2}}} V_{12}^* \, . \tag{A.78}$$

We may construct a basis of independent bilinears consisting of the scalar S_{12}, the three one-forms in (A.75) and the two canonical $SU(2)$ two-forms in (A.77) and (A.78). All other bilinears may be obtained from wedge products of these bilinears. The torsion conditions of the non-basis elements should then be either automatically satisfied by imposing the equations for the basis forms or impose additional algebraic constraints.

Integrability of the torsion conditions (2.119) imply the warp factor satisfies

$$H = -\frac{1}{2}\log[1-|S_{12}|^2] \, . \tag{A.79}$$

We may use it as a coordinate for $\mathrm{Im}\,[S_{12}^* K_{12}]$. Moreover integrability of the torsion conditions implies that the flux $F^{(2)}$ is fixed to be

$$F^{(2)} = -\frac{1}{|S_{12}|^2}\mathrm{d}H \wedge (K_{11} + K_{22}) \, , \tag{A.80}$$

which is easily shown to be both closed and co-closed and therefore $F^{(2)}$ satisfies both its Bianchi identity and its equation of motion. The torsion conditions for K_{11} and K_{22} imply for the Kähler form on $\mathcal{M}_4$ that

$$j = -\frac{\mathrm{e}^{-2H}}{4m}\mathrm{d}\left(\mathrm{e}^{2H}(K_{11} - K_{22})\right) \, , \tag{A.81}$$

which is conformally closed. In light of this we define the rescaled real and complex two forms

$$J = m^2 \mathrm{e}^{2H} j \, , \qquad \Omega = m^2 \mathrm{e}^{3H} V_{12}^* \, , \tag{A.82}$$

for the resulting four-fold as $\widetilde{M}_4$, which satisfy

$$\mathrm{d}J = 0 \, , \qquad \bar{\mathcal{D}}\Omega = -\frac{3\mathrm{i}m}{2}\mathrm{e}^{2H}(K_{11} - K_{22})\wedge \Omega \, . \tag{A.83}$$

From (2.7) we see that P is a $(1,0)$ form with respect to the induced complex structure defined by J. The metric after this redefinition takes the form

$$\mathrm{d}s^2(\mathcal{M}_7) = \frac{\mathrm{e}^{2H}}{4}(K_{11}-K_{22})^2+\frac{1}{4(1-\mathrm{e}^{-2H})}(K_{11}+K_{22})^2+\frac{\mathrm{e}^{-2H}}{m^2(1-\mathrm{e}^{-2H})}\mathrm{d}H^2+\frac{\mathrm{e}^{-2H}}{m^2}\mathrm{d}s^2(\widetilde{\mathcal{M}}_4) \tag{A.84}$$

where $\mathrm{d}s^2(\widetilde{\mathcal{M}}_4)$ is Kähler. As K_{11} and K_{22} are Killing vectors so are the linear combinations

$$K = K_{11} - K_{22} \ , \quad L = K_{11} + K_{22} \ , \tag{A.85}$$

and they satisfy the algebraic conditions

$$||K||^2 = 4\mathrm{e}^{-2H} \ , \quad ||L||^2 = 4(1-\mathrm{e}^{-2H}) \ , \quad K_\mu L^\mu = 0 \tag{A.86}$$

and the differential equations

$$\mathrm{d}(\mathrm{e}^{2H}K) = -\frac{4}{m}J \ , \qquad \mathrm{d}\left(\frac{1}{1-\mathrm{e}^{-2H}}L\right) = 0 \ . \tag{A.87}$$

Decompactification to AdS$_5$

We may introduce local coordinates adapted to these two Killing directions as

$$K^\# = m\frac{\partial}{\partial\psi} \ , \quad L^\# = m\frac{\partial}{\partial\varphi} \ , \tag{A.88}$$

with dual one-forms

$$K = \frac{4}{m}\mathrm{e}^{-2H}\left(\mathrm{d}\psi + \frac{1}{2}\rho\right) \ , \quad L = \frac{4}{m}(1-\mathrm{e}^{-2H})(\mathrm{d}\varphi + \sigma) \ . \tag{A.89}$$

The one-forms ρ and σ are both independent of ψ and φ. From (A.87) we see that σ is closed and therefore locally exact and may be set to zero by a local change of coordinates. The metric takes the form

$$m^2\mathrm{d}s^2(\mathcal{M}_7) = \frac{\mathrm{e}^{-2H}}{1-\mathrm{e}^{-2H}}\mathrm{d}H^2+4(1-\mathrm{e}^{-2H})\mathrm{d}\varphi^2+\mathrm{e}^{-2H}\left(4\left(\mathrm{d}\psi+\frac{1}{2}\rho\right)^2+\mathrm{d}s^2(\widetilde{\mathcal{M}}_4)\right) \ . \tag{A.90}$$

These explicit coordinates induce a splitting of the exterior derivative as

$$\mathrm{d} \to \mathrm{d}\varphi\frac{\partial}{\partial\varphi} + \mathrm{d}H\frac{\partial}{\partial H} + \mathrm{d}\psi\frac{\partial}{\partial\psi} + \mathrm{d}_4 \ . \tag{A.91}$$

With this splitting equation (A.83) decomposes as

$$\partial_\varphi \Omega = \partial_H \Omega = 0 \ , \tag{A.92}$$

$$\partial_\psi \Omega = -6\mathrm{i}\Omega \ , \tag{A.93}$$

$$\bar{\mathcal{D}}_4 \Omega = -\frac{3\mathrm{i}}{2}\rho \wedge \Omega \ . \tag{A.94}$$

Equation (A.93) may be solved by extracting a phase from Ω. Equation (A.94) implies that the Ricci form on $\widetilde{\mathcal{M}}_4$ is

$$\mathfrak{R} = 6J - \mathrm{d}Q \ . \tag{A.95}$$

Combining these terms, the full $10d$ metric is

$$\begin{aligned} \mathrm{d}s^2 =& \mathrm{e}^{2H}\left(\mathrm{d}s^2(\mathrm{AdS}_3) + \frac{\mathrm{e}^{-2H}}{m^2(1-\mathrm{e}^{-2H})}\mathrm{d}H^2 + \frac{4(1-\mathrm{e}^{-2H})}{m^2}\mathrm{d}\varphi^2\right) + \frac{1}{m^2}\Big[\ (2\mathrm{d}\psi+\rho)^2 + \mathrm{d}s^2(\widetilde{\mathcal{M}}_4)\Big] \\ =& \mathrm{d}s^2(\mathrm{AdS}_5) + \frac{1}{m^2}\Big[\ (2\mathrm{d}\psi+\rho)^2 + \mathrm{d}s^2(\widetilde{\mathcal{M}}_4)\Big] \ . \end{aligned} \tag{A.96}$$

The first term in the brackets with the warp factor included is in fact the metric on AdS_5 with Ricci-tensor satisfying $R_{\mu\nu} = -4m^2 g_{\mu\nu}$.

Of course the five-form flux needs to be quantized through the unique five-cycle in the ten-dimensional geometry i.e. we must impose

$$\frac{1}{(2\pi\ell_{\mathrm{s}})^4 g_{\mathrm{s}}}\int_{\mathcal{M}_5^\tau} F \in \mathbb{Z}\,, \tag{A.97}$$

which with the above form for the flux results in

$$\frac{1}{(2\pi\ell_{\mathrm{s}})^4 g_{\mathrm{s}}}\int_{\mathcal{M}_5^\tau}\frac{4}{m^4}\mathrm{dvol}(\mathcal{M}_5^\tau) = \frac{4\mathrm{vol}(\mathcal{M}_5^\tau)}{(2\pi m\ell_{\mathrm{s}})^4 g_{\mathrm{s}}} = N \ . \tag{A.98}$$

The integer N is interpreted as the number of D3-branes as usual. From this it follows straightforwardly that the leading order holographic central charge of the dual 4d SCFT is given by

$$a^{4d} = \frac{\pi}{8G_N^{(10)}}\int_{\mathcal{M}_5^\tau}\mathrm{e}^{3\Delta}\mathrm{dvol}(\mathcal{M}_5^\tau) = \frac{N^2\pi^3}{4\mathrm{vol}(\mathcal{M}_5^\tau)}\,, \tag{A.99}$$

exactly as in the constant τ, Sasaki–Einstein case.

Due to the relation (2.126), the volume now receives corrections with respect to the constant τ case. In particular, using $6J_4 = \mathrm{d}Q + \mathfrak{R}_4$, we have

$$\mathrm{vol}(\mathcal{M}_5^\tau) = \int_{\mathcal{M}_5^\tau} (\mathrm{d}\psi + \sigma) \wedge \frac{J_4 \wedge J_4}{2} = \frac{\pi^3 \ell}{9} \int_{\mathcal{M}_4} c_1(\mathcal{M}_4)^2 - 2c_1(\mathcal{M}_4) \wedge c_1(\mathcal{L}_D) + c_1(\mathcal{L}_D)^2 \ , \tag{A.100}$$

where the first term is the result of the volume for quasi-regular Sasaki–Einstein manifolds. Here $\mathcal{L}_D$ is the duality bundle defined with the connection (1.8), which encodes the varying axio-dilaton.

Appendix B

Central charges and properties of Calabi–Yau's

B.1 Supergravity Central Charges

In this appendix we give details on the formulae used to compute the holographic central charges.

B.1.1 Holographic Central Charges at Leading Order

The leading order term in the central charge is given by the Brown-Henneaux formula [72]

$$c_{\text{sugra}} = \frac{3}{2mG_N^{(3)}}\,, \tag{B.1}$$

where $G_N^{(3)}$ is the three dimensional Newton constant obtained by the reduction of the Type IIB/11d supergravity action on the internal manifold. The relevant part of the action in dimension d is

$$S_d = \frac{1}{16\pi G_N^{(d)}}\int_{M_d} *_d R^{(d)}\,. \tag{B.2}$$

We are interested in dimension $D = 10/11$ warped backgrounds of the form

$$\mathrm{d}s^2(M_D) = \mathrm{e}^{2H}\mathrm{d}s^2(\mathrm{AdS}_3) + \mathrm{d}s^2(M_{D-3})\,, \tag{B.3}$$

where H a function of the internal manifold only. In this background the action in (B.2) can be expressed as

$$S_D = \frac{1}{16\pi G_N^{(D)}}\int_{M_{D-3}} \mathrm{e}^{H}(*_{D-3}1)\int_{\mathrm{AdS}_3} *_3(R^{(3)} + ...)\,. \tag{B.4}$$

This leading order piece is exactly the action (B.2) in three dimensions. From this we identify the $d = 3$ Newton constant to be

$$\frac{1}{G_N^{(3)}} = \frac{1}{G_N^{(D)}} \int_{M_{D-3}} \mathrm{e}^H \mathrm{dvol}(M_{D-3}) \,, \tag{B.5}$$

and hence

$$c_{\mathrm{sugra}} = \frac{3}{2mG_N^{(D)}} \int_{M_{D-3}} \mathrm{e}^H \mathrm{dvol}(M_{D-3}) \,. \tag{B.6}$$

In 10d the Newton's constant is $G_N^{(10)} = 2^3\pi^6\ell_s^8$, whilst in 11d it is given by $G_N^{(11)} = 2^4\pi^7{\ell_{\mathrm{p}}}^9$.

B.1.2 Holographic Central Charges at Sub-leading Order

We may compute the sub-leading order terms in 11d supergravity by making use of the X_8 anomaly inflow polynomial [84] and the relation

$$\begin{aligned} c_L - c_R &= 96\pi\beta \,, \\ S_{CS} &= \beta \int_{\mathrm{AdS}_3} \omega_{CS}(\Gamma) \subset S_{3d} \,, \end{aligned} \tag{B.7}$$

as found in [82]. We reduce the 11d Chern-Simons term

$$S_{CS} = -\frac{(4\pi\kappa_{11})^{2/3}}{2\kappa_{11}^2} \int_{M_{11}} C_3 \wedge X_8 \tag{B.8}$$

on the internal space, with X_8 given by

$$X_8 = \frac{1}{(2\pi)^4 2^6 \cdot 3} \left(\mathrm{Tr}[\mathcal{R}^4] - \frac{1}{4}(\mathrm{Tr}[\mathcal{R}^2])^2 \right) \,, \tag{B.9}$$

and

$$\begin{aligned} \mathcal{R}^a{}_b &= \frac{1}{2} R^a{}_{b\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,, \\ 2\kappa_{11}^2 &= (2\pi)^8 {\ell_{\mathrm{p}}}^9 \,. \end{aligned} \tag{B.10}$$

Integrating (B.8) by parts we have

$$S_{CS} = \frac{(4\pi\kappa_{11})^{2/3}}{2\kappa_{11}^2} \int_{M_{11}} G_4 \wedge X_7 \,, \tag{B.11}$$

where $X_8 \equiv \mathrm{d}X_7$. In our solution the internal eight-dimensional space is $S^2 \times Y_3$, and given the form of the G_4 flux (3.74), we have that $X_7 = \frac{1}{3(2\pi)^4 2^7} \omega_{CS}(\Gamma_{\mathrm{AdS}_3}) \wedge \mathrm{Tr}[\mathcal{R}_{Y_3}^2]$

so that we determine

$$\beta = \frac{\mathrm{e}^{4H}}{3(2\pi\ell_\mathrm{p})^3 m(2\pi)^4 2^7} \int_{Y_3} J_{Y_3} \wedge \mathrm{Tr}[\mathcal{R}^2_{Y_3}] \,. \tag{B.12}$$

B.2 Properties of Kähler and Calabi–Yau Varieties

In this appendix we collect some essential theorems related to the elliptically fibered Calabi–Yau threefolds that we consider as our compactification spaces.

B.2.1 Elliptic Fibrations

In the following it will be useful to have some geometric basics about elliptic fibrations in place for studying F-theory solutions. Here our main interest is in elliptic threefolds, but much can be generalised to other dimensions. We consider Calabi–Yau threefolds Y_3, which are elliptically fibered over a base B, which is a complex surface. Denote the projection map $\pi : Y_3 \to B$. Furthermore we assume there is a section, which as explained earlier implies the existence of a Weierstrass model.

It will be very important in the following to determine the possible divisors (4d submanifolds) in such a geometry, which is the content of the Shioda-Tate-Wazir theorem [134], which implies that the divisors of an elliptic Calabi–Yau threefold Y_3 with a section, fall into the following three classes:

1. Section: This is the divisor obtained by the image of the base B in Y_3. We denote it simply by B. The dual $(1,1)$-form will be denoted by ω_0. [1]

2. Pull-back of curves in the base B: For every effective curve $C_\alpha \in H_2(B)$ we have a divisor in Y_3 given by $\pi^*(C_\alpha)$. We will refer to these as $\widehat{C}_\alpha \equiv \pi^*(C_\alpha)$, and denote the dual $(1,1)$-forms by ω_α.

3. Resolution/Cartan divisors: These divisors occur whenever there is a singularity in the Weierstrass model of the elliptic fibration, and they are given in terms of rational curves, that are obtained from the resolution of the singularities, fibered over a curve in the base (which are components of the discriminant). In the literature these are often referred to as Cartan divisors, as they are (in many cases) labeled by the simple roots of the Lie algebra associated to the Kodaira singular fiber. The Cartan divisors will be denoted by D_i, and the dual $(1,1)$-forms by ω_i.

[1] An elliptic fibration can have more rational sections, in which case there are additional divisors and $(1,1)$ forms, which generate the Mordell-Weil group of the fibration. As this will not play any role here, we refrain from discussing these further.

For the most part of this thesis we will consider smooth Weierstrass fibrations, *i.e.* there are no Cartan divisors. However this can be easily generalised. Divisors are dual to $(1,1)$-forms, and the Shioda-Tate-Wazir theorem thus implies that the Kähler form of the Calabi–Yau can be expanded as

$$J_{Y_3} = k_0\omega_0 + \sum_\alpha k_\alpha\omega_\alpha + \sum_i k_i\omega_i\,. \tag{B.13}$$

We will require that the Kähler class of the base

$$J_B = \sum_\alpha k_\alpha\omega_\alpha\,, \tag{B.14}$$

is dual inside B to a curve C, implying that $k_\alpha \in \mathbb{Z}^+$. This means that J_B is in fact the Kähler class associated to the Hodge metric on B [69]. However, we do not require any such integrality for k_0.

The non-trivial triple intersections of the basis $\omega_I = \{\omega_0, \omega_\alpha, \omega_i\}$ in the Calabi–Yau

$$C_{IJK} = D_I \cdot_{Y_3} D_J \cdot_{Y_3} D_K = \int_{Y_3} \omega_I \wedge \omega_J \wedge \omega_K\,, \tag{B.15}$$

can be evaluated in terms of data of the base B as follows[2]

$$\begin{aligned} C_{000} &= \int_B c_1(B)^2 = 10 - h^{1,1}(B) \\ C_{00\alpha} &= -c_1(B)\cdot C_\alpha \\ C_{0\alpha\beta} &= Q_{\alpha\beta} \\ C_{\alpha ij} &= -\mathcal{C}_{ij} Q_{\alpha\beta} C^\beta\,, \end{aligned} \tag{B.16}$$

where $Q_{\alpha\beta}$ is the intersection form on B and $\mathcal{C}_{ij}$ the Cartan matrix of the gauge algebra $\mathfrak{g}$ associated to the singularity. The triple intersection C_{ijk} were determined in [135–137] and depend on codimension two singularities, which are labeled by representations of $\mathfrak{g}$.

In deriving these intersection numbers we have made use of the intersection relation in Y_3[3]

$$\sigma \cdot_{Y_3} (\sigma + c_1(B)) = 0\,. \tag{B.17}$$

We will also need to compute intersections with $c_2(Y_3)$. The total Chern class for the Calabi–Yau can be written as

$$c(Y_3) = (1+[w])(1+[x])(1+[y])\frac{c(B)}{(1+[y^2])}\,, \tag{B.18}$$

[2]Whenever we write $\cdot$ without any subscript in the following, this will denotethe intersection in B, unless otherwise stated.

[3]This follows from the fact that w, x, y cannot vanish at the same time, and thus $[w]\cdot[x]\cdot[y] = 0$ as intersections in X_4, or noting that the class of the hypersurface (1.15) is $[y^2]$, this becomes $\sigma\cdot(\sigma + c_1(B))\cdot[y^2] = 0$.

where $c(B)$ is the total Chern class of the base and the denominator corresponds to the class of the hypersurface equation (1.15). Expanding this to second order we obtain[4]

$$c_2(Y_3) = c_2(B) + 11c_1(B)^2 + 12\omega_0 \wedge c_1(B)\,, \tag{B.19}$$

which allows the computation of integrals over $c_2(Y_3)$ using the intersection numbers in (B.16).

Throughout our considerations we will assume the base B to be smooth as a variety, albeit the induced metric on B will have singularities that we will discuss in some detail later on.

B.2.2 Useful Relations

First let Y be a Kähler manifold with a given Kähler metric, $g_{\mu\bar{\nu}}$. Then the Kähler form associated to this metric is

$$J = \mathrm{i} g_{\mu\bar{\nu}} \mathrm{d}z^{\mu} \wedge \mathrm{d}\bar{z}^{\bar{\nu}}\,, \tag{B.20}$$

which is a closed $(1,1)$-form that is a representative of the cohomology class known as the Kähler class; where it would be otherwise unambiguous we shall abuse notation and use J to refer to both the explicit representative and the class. As J is formed from the Kähler metric then it is real and positive. This means that

$$\int_C J > 0\,, \quad \int_S J \wedge J > 0\,, \quad \cdots\,, \tag{B.21}$$

where C is any curve in Y, S any surface, and so on. One can find a summary of this standard information in, for example, [138]. Further, it is known that a compact complex manifold admits an holomorphic embedding into projective space if and only if it admits a Kähler metric whose associated Kähler form is an integral class [139]. As a corollary to Yau's theorem, any compact strict Calabi–Yau, Y_n, of dimension $n \geq 3$ can be embedded as a complex submanifold of a complex projective space, and thus we can conclude that any Calabi–Yau threefold permits an integral Kähler class.

After these introductory remarks we now collect several useful formulas. For this we will specialise to the case of elliptically fibered Calabi–Yau threefolds as in section 1.1.2, with base B, which is a Kähler surface. Various properties of the base B will feature in the main text, in particular relations for topological invariants such as

$$3\sigma(B) + 2\chi(B) = \int_B c_1(B)^2\,, \tag{B.22}$$

where $\sigma(B)$ is the signature of the manifold and $\chi(B)$ is the Euler number. In terms

[4] Here we used the relation (B.17), which holds on Y_3.

of the Hodge numbers of B these can be written as

$$\begin{aligned}\chi(B) &= 2 - 4h^{0,1}(B) + 2h^{0,2}(B) + h^{1,1}(B)\\ \sigma(B) &= (2h^{0,2}(B)+1) - (h^{1,1}(B)-1) = b_2^+ - b_2^- \,,\end{aligned} \tag{B.23}$$

where $b_2^\pm$ are the number of self-dual and anti-self-dual two-forms of B. So far we have only assumed that B is a compact Kähler surface.

Now let us further suppose that B is the base of an elliptic fibration $\pi : Y_3 \to B$ with section. As explained in the main text this does restrict the type of Kähler surfaces that can function as B. In particular, the existence of the section implies that

$$\pi_1(B) = 0 \qquad \Longrightarrow \qquad h^{0,1}(B) = 0\,. \tag{B.24}$$

Furthermore the elliptic fibration must be Calabi–Yau which means that

$$h^{0,2}(B) = 0\,, \tag{B.25}$$

as otherwise any $(0,2)$ forms on B would give rise to $(0,3)$ forms on the Calabi–Yau. Summarising, if B is the base of an elliptically fibered Calabi–Yau threefold then

$$\int_B c_1(B)^2 = 3\sigma(B) + 2\chi(B) = 10 - h^{1,1}(B)\,. \tag{B.26}$$

This agrees with the results given in [86], and is a general result for any base B which may support a non-trivial Calabi–Yau elliptic fibration over it.

We will require in the main text to determine the second Chern class of the Calabi–Yau threefold when integrated over an arbitrary divisor P of Y_3. We have that

$$\int_P c_2(Y_3) = \int_P (c_2(P) - c_1(P)^2) = 2(h^{1,1}(P) - 4h^{0,2}(P) + 2h^{0,1}(P) - 4)\,, \tag{B.27}$$

where the first equality follows via adjunction. As we can see the integral over the second Chern class over any divisor is always an even integer.

B.2.3 Ample Divisors in Elliptically Fibered Calabi–Yau Threefolds

We shall now collect results about the ampleness properties of divisors in an elliptically fibered Calabi–Yau threefold. An M5-brane wrapping a divisor D will only have an AdS dual when D is ample, as the divisor must be dual to a $(1,1)$-form in the Kähler cone of the Calabi–Yau, following from the 11d supergravity solution in section 3.3.

First we shall be general and consider Y any smooth algebraic variety, with D a divisor on Y. The Nakai–Moishezon [140,141] criterion for ampleness (see *e.g.* [142] for an in depth discussion) is that

$$D^{\dim(X)} \cdot X > 0\,, \tag{B.28}$$

for every closed subvariety X in Y. We remark that since the Nakai–Moishezon criterion is just the intersection theory dual of the statement that

$$\int_X \omega^{\dim(X)} > 0\,, \tag{B.29}$$

where ω is the dual $(1,1)$-form to the divisor D; in this way we can see that every ample divisor is dual to a $(1,1)$-form inside of the Kähler cone of Y.[5]

With this in hand we shall now specifically consider a smooth elliptically fibered Calabi–Yau threefold, $\pi : Y_3 \to B$, and the ampleness of the divisors thereon. It was described in section B.2.1 that an elliptic fibration, with trivial Mordell–Weil group, has three distinct classes of divisors which span the Néron–Severi lattice of divisors of Y_3. These are the zero-section, which provides a copy of B in the fiber, the pullbacks of the curves in the base, $\widehat{C}_\alpha = \pi^*(C_\alpha)$, and the Cartan divisors associated to the resolution of singularities, D_i. We will be interested in the triple intersection numbers of these divisors. The triple intersection numbers that are of interest to us were determined in [86], and were recapped in (B.16).

Let us first consider a smooth Weierstrass model Y_3, where we recall that there are no resolution divisors, we consider a divisor in the linear system

$$D \in |MB + N\widehat{C}|\,. \tag{B.30}$$

We are interested in knowing for what values of $M, N \geq 0$ is this divisor not ample. We know from the Nakai–Moishezon criterion for ampleness that

$$D \cdot \Sigma > 0\,, \tag{B.31}$$

for every curve Σ in Y_3, which includes the curve C in B which $\widehat{C}$ is the pullback of, *i.e.* $\widehat{C}$ is the elliptic surface obtained by restricting the fibration to C. We can then compute

$$D \cdot C = MB \cdot C + N\widehat{C} \cdot C = MB \cdot B \cdot \widehat{C} + NB \cdot \widehat{C} \cdot \widehat{C}\,, \tag{B.32}$$

where in the final equality we have used that

$$C = B \cdot \widehat{C}\,. \tag{B.33}$$

[5] A subset of the ample divisors consists of the *very* ample divisors, which are those divisors which are linearly equivalent to the hyperplane class of a projective embedding of Y [143].

Using the triple intersection numbers listed in (B.16), along with adjunction,

$$c_1(B) \cdot C = C \cdot C + 2 - 2g\,, \tag{B.34}$$

we can see that there is the constraint

$$D \cdot C = (N - M) C \cdot C + M(2g - 2) > 0\,. \tag{B.35}$$

For $N \gg M$, this is equivalent to the statement that D is not ample in Y_3 if C is not ample in B. It is also clear from this formula that, for example, when $M = N$ we need we consider an elliptic surface $\widehat{C}$, where the base curve C is such that

$$g \geq 2\,, \tag{B.36}$$

and ampleness clearly implies a non-trivial interdependence between M, N, and g. Further one would like to determine whether there are constraints on ampleness when $M = 0$. While the constraint (B.35) only requires that C must have a strictly positive self-intersection in the base we further note that the Nakai–Moishezon criterion for ampleness requires also that the triple-intersection of the divisor in Y_3 be strictly positive. For an elliptic surface we observe that

$$\widehat{C} \cdot \widehat{C} \cdot \widehat{C} = 0\,, \tag{B.37}$$

as was evidenced directly from the Hodge numbers in (3.154), and thus we determine that when $M = 0$ the divisor cannot be ample.

For the case that is not a smooth Weierstrass model we can consider a divisor in the linear system

$$D \in |MB + N\widehat{C} + M_i D_i|\,, \tag{B.38}$$

and consider again $D \cdot C$, however we should not include in this sum the Cartan divisor associated to the affine node of the Dynkin diagram as it is not an independent divisor inside the Neron–Severi lattice, and so the M_0 will not be a free parameter. We can see again from the triple intersection numbers (B.16) that

$$D_i \cdot B \cdot \widehat{C}\,, \tag{B.39}$$

is only non-zero when D_i is precisely the divisor associated to the affine node, and so the same conclusion on the constraints on M, N will hold as in the smooth Weierstrass case.

Finally one can study the case where C is a smooth rational curve of self-intersection $C \cdot C = -n$ for $n = 3, \cdots, 12$, excepting $n = 9, 10, 11$. These setups involve C not being ample in B, and correspond to the non-Higgsable clusters [73]. The self-intersection of the curve in the base is severe enough that it mandates a total space singularity above that curve in the Weierstrass model, with a specific kind of singular fiber located above the curve C depending on n. In such a setup one can again compute

$$D \cdot C = (n-2)M - nN > 0\,, \tag{B.40}$$

which, given that n is a positive integer generally requires that $M > N$, if the divisor D is to be ample.

Appendix C

$(0,2)$ solutions appendix

C.1 Details for the Baryonic Twist Solution

In this appendix we provide some more details on the baryonic twist solution of section 4.1.2. The solution to (4.4) that we shall use was found in [57] for general values of s, here we are interested in the $s = 2$ case[1]. This was later discussed in [56], where it was interpreted as a Type IIB solution of the form $\mathrm{AdS}_3 \times T^2 \times \mathcal{M}_5^\tau$, with the regularity analysis performed therein. As we show here, the same solution to (4.4) yields an F-theory geometry of the form $\mathrm{AdS}_3 \times \mathrm{K3} \times \mathcal{M}_5^\tau$, with the same manifold (in particular the same metric) $\mathcal{M}_5^\tau$. After reviewing the derivation of the local form of the solution, for completeness, we shall perform a similar analysis of the regularity and global properties, with some minor changes from [56].

The starting point is a cohomegenity one ansatz for the Kähler metric on $\mathcal{M}_4$,

$$\mathrm{d}s^2(\mathcal{M}_4) = \frac{\mathrm{d}r^2}{U(r)} + U(r)r^2\left(\mathrm{d}\varphi + \frac{1}{2}\cos\theta\mathrm{d}\chi\right)^2 + \frac{r^2}{4}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\chi^2) \ , \qquad \text{(C.1)}$$

for which, after changing variable to $x = 1/r^2$, one can find the explicit solution to (2.36) as

$$U(x) \ = \ 1 - a(x-1)^2 \qquad \text{(C.2)}$$

depending on one integration constant a.

[1] In the notation of [57] $s = n+1$. The authors of [57] were mainly interested in the cases $s = 3$ and $s = 4$.

C.1.1 The local F-theory geometry

The 7d part of the metric takes the form

$$\begin{aligned} m^2 \mathrm{d}s^2(\mathcal{M}_7) &= \frac{1}{4}\left(\mathrm{d}\psi - 2a(x-1)D\phi\right)^2 \\ &+ a\left[\frac{\mathrm{d}x^2}{4x^2U} + UD\varphi^2 + \frac{1}{4}\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\chi^2\right)\right) + x\mathrm{d}s^2(B_2)\right] \ , \end{aligned} \tag{C.3}$$

where $D\varphi = \mathrm{d}\varphi + \frac{1}{2}\cos\theta\mathrm{d}\chi$. Using (2.54) we can read off the expression for the flux

$$mF^{(2)} = -\frac{1}{2ax^2}\mathrm{d}\psi\wedge\mathrm{d}x - 2\mathrm{dvol}(B_2) - \frac{1}{2}\mathrm{dvol}(S^2) \ . \tag{C.4}$$

The regularity analysis performed in the next subsection shows that the base of this local $U(1)$ fibration is itself not a manifold[2], instead a change of coordinates is useful to describe the global geometry and results in the solution in the form presented in (5.32) and (4.45). The profile of the axio-dilaton is determined (implicitly) by the condition that the metric on $\mathcal{Y}_4$ is a Ricci-flat metric, and thus $\mathcal{Y}_4$ is an elliptically fibered K3, with base $B_2 = \mathbb{P}^1$. Note that we will not determine explicitly the metric on $\mathbb{P}^1$ and in particular this cannot be the Einstein metric, but the stringy-cosmic string metric of [12], induced by the elliptic fibration. In particular, the metric will have singularities at the discriminant loci. We thus continue to distinguish the two two-spheres in the geometry by referring to them as S^2 and $\mathbb{P}^1$, respectively.

At this stage the background depends on two arbitrary constants m, a and we now determine which values of these allow for a globally defined solution.

C.1.2 Regularity

We first consider regularity of the metric and later address the quantisation of the flux (similar discussions have appeared in [32, 56]). The metric on $\mathcal{M}_7$ is

$$\begin{aligned} \mathrm{d}s^2(\mathcal{M}_7) =& \frac{1}{4}\left(\mathrm{d}\psi - 2a(1-x)\left(\mathrm{d}\varphi + \frac{1}{2}\cos\theta\mathrm{d}\chi\right)\right)^2 \\ &+ a\left(\frac{\mathrm{d}x^2}{4x^2U} + U\left(\mathrm{d}\varphi + \frac{1}{2}\cos\theta\mathrm{d}\chi\right)^2 + \frac{1}{4}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\chi^2) + x\mathrm{d}s^2(B_2)\right) \ . \end{aligned} \tag{C.5}$$

We require that the warp factor does not vanish and therefore the range of the coordinate x cannot include $x = 0$. This implies that the 7d geometry is topologically $\mathcal{M}_5^\tau \times \mathbb{P}^1$, with $\mathcal{M}_5^\tau$ the five-dimensional space defined by $x =$ constant, and therefore we need only analyse the regularity of $\mathcal{M}_5^\tau$, subject to x avoiding $x = 0$. The range

[2] A similar situation occurs with the Sasaki–Einstein $Y^{p,q}$ manifolds. The Kähler base of $Y^{p,q}$ in the canonical Sasaki–Einstein coordinates is not in general a manifold.

of x is fixed to lie between the two roots of $U(x)$

$$x_\pm = 1 \pm \frac{1}{\sqrt{a}} \ . \tag{C.6}$$

Clearly to avoid $x = 0$ it is necessary to have $x_- > 0$, so that $a > 1$, and it follows that $U(x)$ is positive between the two roots for all values $a > 1$.

The Base $\mathcal{Z}_4$

Let us first consider the four-dimensional part of the metric, namely the Kähler base $\mathcal{M}_4$. The round S^2 appearing in $\mathcal{M}_4$ has coordinates θ and χ with the canonical coordinate periodicities $\theta \in [0, \pi]$ and $\chi \in [0, 2\pi]$. Near to the zeroes of U at $x = x_\pm$, the degenerating part of the metric is

$$\frac{1}{x_\pm^2 U'(x_\pm)} \left(\mathrm{d}\rho^2 + (x_\pm U'(x_\pm))^2 \rho^2 \mathrm{d}\varphi^2 \right) \ , \tag{C.7}$$

where $\rho = 2\sqrt{x - x_\pm}$, respectively. For this to be locally $\mathbb{R}^2$ at both end-points it is necessary that $x_\pm U'(x_\pm)$ has the same value at both roots; a trivial calculation shows this is not the case and therefore there is no choice of periodicity of φ that gives a smooth metric.

As in [32], the way to proceed is to show that one can still view the five-dimensional space as a circle fibration over a base $\mathcal{Z}_4$, albeit one with metric different from the local Kähler metric on $\mathcal{M}_4$. Changing coordinates from (ψ, φ) to (α, ϕ) by $\alpha = \psi$, $\phi = 2\varphi + \psi$, the 5d metric takes the form

$$\begin{aligned} \mathrm{d}s^2(\mathcal{M}_5) =& \frac{w(x)}{4} \left(\mathrm{d}\alpha \ + g(x)(\mathrm{d}\phi + \cos\theta \mathrm{d}\chi) \right)^2 \\ &+ \frac{a}{4} \left[\frac{\mathrm{d}x^2}{x^2 U} + \frac{U}{w} (\mathrm{d}\phi + \cos\theta \mathrm{d}\chi)^2 + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\chi^2 \right] \ . \end{aligned} \tag{C.8}$$

With this change of coordinates we may avoid potential conical singularities at the endpoints of x if ϕ has period 2π due to the remarkable fact that

$$\frac{(U'(x_\pm) x_\pm)^2}{w(x_\pm)} = 1 \ . \tag{C.9}$$

As in [32] we may introduce a new angular coordinate defined by

$$\cos\zeta = -\frac{1 + a(x-1)}{\sqrt{w}} \ , \qquad \sin\zeta = \frac{\sqrt{aU}}{\sqrt{w}} \ , \tag{C.10}$$

with $\zeta \in [0, \pi]$. Performing this change of coordinates is not particularly useful and so we shall keep the x coordinates in the following.

At fixed x between the two roots the base $\mathcal{Z}_4$ with metric given in the second line of (C.8), is a circle bundle over the round two-sphere, where the $U(1)$ fiber coordinate is ϕ. The Chern number of this bundle is obtained by computing the integral of the curvature two-form of the connection on the $U(1)$ and gives

$$\frac{1}{2\pi}\int_{S^2} \mathrm{d}(-\cos\theta \mathrm{d}\chi) = 2 \ . \tag{C.11}$$

This identifies the three-dimensional space at fixed x to be $S^3/\mathbb{Z}_2$. Furthermore it follows that the four-dimensional base $\mathcal{Z}_4$ has topology $S^2 \times S^2$. For the following it is useful to have an explicit basis for the homology group $H_2(\mathcal{Z}_4;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$. The two natural two-cycles are the two S^2's, whose cycles we denote by C_1, C_2 in keeping with the notation in [32]. Since the metric on $\mathcal{Z}_4$ is not a product metric the location of the two S^2's is not clear, but we may take C_1 to be the fiber S^2 at fixed θ, χ on the round two-sphere. There are two two-cycles which are visible in the geometry; namely the two S^2's at the south and north poles of the fiber S^2 (at $\zeta = 0, \pi$ respectively or equivalently $x = x_-, x_+$), let us call them S_1 and S_2. Then the two-cycles C_i are

$$2C_1 = S_1 - S_2 \ , \quad 2C_2 = S_1 + S_2 \ , \tag{C.12}$$

with dual cohomology elements

$$\begin{aligned}
\omega_1 &= -\frac{1}{4\pi}\left(\sin\zeta \mathrm{d}\zeta\wedge(\mathrm{d}\phi + \cos\theta \mathrm{d}\chi) - \cos\zeta\sin\theta \mathrm{d}\theta\wedge \mathrm{d}\chi\right) \ , \\
\omega_2 &= \frac{1}{4\pi}\sin\theta \mathrm{d}\theta\wedge \mathrm{d}\chi \ ,
\end{aligned} \tag{C.13}$$

satisfying

$$\int_{C_i}\omega_j = \delta_{ij} \ . \tag{C.14}$$

As we wish to be precise in comparing the geometry here with that of the $Y^{p,q}$ manifolds, we will perform some additional checks on the base $\mathcal{Z}_4$.

The Euler characteristic of a four-manifold $\mathcal{M}_4$ may be computed by using the Chern-Gauss-Bonnet theorem

$$\chi(\mathcal{M}_4) = \frac{1}{32\pi^2}\int_{\mathcal{M}_4}\sqrt{g}\left(|W|^2 - 2\left|\mathrm{Ric} - \frac{R}{4}g\right|^2 + \frac{1}{6}R^2\right) \ , \tag{C.15}$$

where norms are computed using the metric, W denotes the Weyl tensor and Ric the Ricci tensor. Computing this for our metric we find[3]

$$\chi(\mathcal{Z}_4) = 4 \ . \tag{C.16}$$

[3] In general, for the product of two Riemann surfaces $\Sigma_1\times\Sigma_2$ of genus g_1, g_2, respectively, we have $\chi(\Sigma_1\times\Sigma_2) = 4(1-g_1)(1-g_2)$.

We may compute the signature using the Hirzebruch signature theorem

$$\sigma(\mathcal{M}_4) = \frac{1}{48\pi^2} \int_{\mathcal{M}_4} \sqrt{g}(|W^+|^2 - |W^-|^2) \tag{C.17}$$

and indeed we find[4]

$$\sigma(\mathcal{Z}_4) = 0 \ . \tag{C.18}$$

Let us also check that $\mathcal{Z}_4$ is a complex manifold. To do so we compute the exterior derivative of the associated $(0,2)$ two-form. As is well-known the exterior derivative of the holomorphic n-form on a complex manifold of complex dimension n satisfies

$$\mathrm{d}\Omega = \mathrm{i}\widehat{P} \wedge \Omega \ , \tag{C.19}$$

where $\widehat{P}$ is a one form potential for the Ricci-form $\mathfrak{R}$, that is, $\mathrm{d}\widehat{P} = \mathfrak{R}$. For the metric on $\mathcal{Z}_4$ we have

$$\Omega = \frac{a}{x}\left(\frac{1}{x\sqrt{U}}\mathrm{d}x + \mathrm{i}\frac{\sqrt{U}}{\sqrt{w}}(\mathrm{d}\phi + \cos\theta \mathrm{d}\chi)\right) \wedge (\mathrm{d}\theta + \mathrm{i}\sin\theta \mathrm{d}\chi) \tag{C.20}$$

and upon taking the exterior derivative one finds that the manifold is complex, with the one-form Ricci potential given by

$$\widehat{P} = \left(1 + \frac{1 + a(2(2x-1) - a(x-1)(1 + x(x-3)))}{w^{3/2}}\right)(\mathrm{d}\phi + \cos\theta \mathrm{d}\chi) \ . \tag{C.21}$$

The corresponding Ricci-form $\mathfrak{R} = \mathrm{d}\widehat{P}$ is proportional to the first Chern class of the tangent bundle of the manifold, and may be integrated over the two two-cycles discussed above. We find

$$\frac{1}{2\pi}\int_{S_1} \mathfrak{R} = 0 \ , \qquad \frac{1}{2\pi}\int_{S_2} \mathfrak{R} = 4 \ , \tag{C.22}$$

which implies

$$c_1(C_1) = c_2(C_2) = 2 \ . \tag{C.23}$$

This discussion establishes that in fact $\mathcal{Z}_4$ is complex-diffeomorphic to the Hirzebruch surface $\mathbb{F}_0 = S^2 \times S^2$, exactly as for the 4d base that appeared in the $Y^{p,q}$ construction in [32][5].

[4]The signature of the product of two Riemann surfaces $\Sigma_1 \times \Sigma_2$ vanishes by Rohlin's theorem, as this is the boundary of its handlebody.

[5]Recall that for a Hirzebruch surface $\mathbb{F}_n$, in a basis of $H_2(\mathbb{F}_n;\mathbb{Z})$ with intersection matrix

$$\begin{pmatrix} -n & 1 \\ 1 & 0 \end{pmatrix}$$

we have the following invariants $\chi(\mathbb{F}_n) = 4, \sigma(\mathbb{F}_n) = 0, c_1(C_1) = -n+2, c_1(C_2) = 2$. We have checked that these invariants identify the base manifold B_4 in [32] as $B_4 \simeq \mathbb{F}_0$. We have also

The Circle Fibration

We now turn to the circle fibration. The norm of the Killing vector $\partial/\partial\alpha$ is $w(x)/4$ and this is nowhere vanishing in the range between the zeroes of $U(x)$. In order to get a compact five-dimensional manifold we need the coordinate α to describe an S^1 bundle over $\mathcal{Z}_4$. We then take it to have period

$$0 \leq \alpha \leq 2\pi\ell \ , \tag{C.24}$$

where ℓ parametrises the arbitrariness of the period of α. We may then rescale α by ℓ^{-1} which implies that

$$\ell^{-1}A = \ell^{-1}g(\mathrm{d}\phi + \cos\theta \mathrm{d}\chi) \tag{C.25}$$

should be a connection on a $U(1)$ bundle over $\mathcal{Z}_4 \simeq S^2 \times S^2$. In general such $U(1)$ bundles are completely specified topologically by the gluing on the equators of the two S^2 cycles C_1 and C_2. These are measured by the corresponding Chern numbers in $H_2(S^2;\mathbb{Z}) = \mathbb{Z}$ which we label $\mathfrak{p}$ and $\mathfrak{q}$. These are given by the integrals of the $U(1)$-curvature two-form $\mathrm{d}A/2\pi$ over the two two-cycles which form the basis of $H_2(\mathcal{Z}_4;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. We may choose ℓ such that $\mathfrak{p}$ and $\mathfrak{q}$ are coprime, $(\mathfrak{p},\mathfrak{q}) = 1$. We first check that $\mathrm{d}A$ is a globally defined two-form. At fixed x between the two roots x_-, x_+ we see that $\mathrm{d}A$ is proportional to the "global angular form" on the $U(1)$ bundle with fibre ϕ and is a globally well-defined one-form, therefore so is $\mathrm{d}A$ on a fixed x slice of $\mathcal{Z}_4$. We must also check how the curvature two-form behaves near to the zeroes of U. We find that the only piece that may be troublesome is the term proportional to $\mathrm{d}x \wedge \mathrm{d}\alpha$ near the poles, however the true radial coordinate is $r = (x - x_i)^{1/2}$ and so this term is proportional to the volume form near the fibre poles and thus is well-defined. Consequently $\mathrm{d}A$ is a globally well-defined smooth two-form on $\mathcal{Z}_4$.

Let us now calculate the periods

$$P_i \equiv \frac{1}{2\pi}\int_{C_i} \mathrm{d}A \ . \tag{C.26}$$

The corresponding integrals of $\ell^{-1}\mathrm{d}A/2\pi$ give the Chern numbers $\mathfrak{p}, \mathfrak{q}$, so that we have $P_1 = \ell\mathfrak{p}$ and $P_2 = \ell\mathfrak{q}$. These are most easily found by first computing the integrals of $\mathrm{d}A$ over the two cycles S_i, namely

$$\frac{1}{2\pi}\int_{S_i} \mathrm{d}A = 2g(x_i) \ , \tag{C.27}$$

checked that computing explicitly these for the metric on $\mathbb{F}_1$ found in [33], gives correctly $\chi(\mathbb{F}_1) = 4, \sigma(\mathbb{F}_1) = 0, c_1(C_1) = 1, c_1(C_2) = 2$, where $C_1 = H - E, C_2 = E$.

from which we find

$$P_1 = \frac{2\sqrt{a}}{1-a}\,, \qquad P_2 \;=\; \frac{2a}{1-a} \qquad \Rightarrow \qquad \frac{P_1}{P_2} = \frac{1}{\sqrt{a}} = \frac{\mathfrak{p}}{\mathfrak{q}} \tag{C.28}$$

which implies that

$$a = \frac{\mathfrak{q}^2}{\mathfrak{p}^2}\,, \qquad \ell = \frac{2\mathfrak{q}}{\mathfrak{q}^2-\mathfrak{p}^2}\,, \qquad x_\pm = 1 \pm \frac{\mathfrak{p}}{\mathfrak{q}}\,. \tag{C.29}$$

Recall that the regularity of the metric required that $a > 1$, which implies that the integers $\mathfrak{p}, \mathfrak{q}$ obey

$$0 < \mathfrak{p} < \mathfrak{q}\ , \tag{C.30}$$

for which there is clearly an infinite number of solutions. We have deliberately used a notation as close as possible to [32], and found that topologically the base $\mathcal{Z}_4$ and the circle fibration are formally identical. More precisely, $\mathcal{Z}_4$ here and the base B_4 in [32] are diffeomorphic as complex manifolds, and the two corresponding circle bundles are characterised by a pair of coprime integers. The regularity of the metric here, implies that the Chern numbers $\mathfrak{p}$, $\mathfrak{q}$ characterising the fibration obey an inequality that is *opposite* to those obeyed by the integers p, q in the $Y^{p,q}$ Sasaki–Einstein manifolds, which was $p > q > 0$! We denote the corresponding five-dimensional manifolds as $\mathcal{M}_5 = \mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$.

To summarise, the geometry of the full Type IIB solution is

$$\mathrm{AdS}_3 \times \mathbb{P}^1 \times \mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}\,, \qquad \mathfrak{Y}^{\mathfrak{p},\mathfrak{q}} = S^1 \to \mathbb{F}_0\,, \tag{C.31}$$

where $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ is a circle fibration over $\mathbb{F}_0 = S^2 \times S^2$. Of course the Kähler metric on this $\mathbb{F}_0$ is not the Einstein, direct-product metric on $S^2 \times S^2$.

As already mentioned, the same $\mathcal{M}_5 = \mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ geometry enters in the solutions with constant τ presented in [56]. Indeed, one can show that the global analysis conducted in [56] matches that presented above[6].

C.1.3 Toric Geometry of $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$

The fact that the manifolds $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ are not Sasaki–Einstein[7] leads to the cones constructed over these, $C(\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}})$, not being Calabi–Yau. In fact, the cone over these $\mathcal{M}_5^\tau$ geometries admit an integrable complex structure, but not a symplectic structure. In particular they are not Kähler [103]. However, both the five-dimensional manifolds $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ and their cones $C(\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}})$ admit an isometric (and holomorhic) $\mathbb{T}^3 \simeq U(1)^3$ action. Therefore, on the one hand, the methods from *toric symplectic geometry*

[6] Denoting the integers p, q in [56] as p_{DGK}, q_{DGK}, one has the following identifications $\mathfrak{p} = q_{DGK}$ and $\mathfrak{q} = p_{DGK} + q_{DGK}$.

[7] In fact, they are neither Einstein nor Sasakian. They are not even contact manifolds [111].

employed in [33] cannot be applied here. In particular, we do not have moment maps whose images would determine the convex polyhedral cones underlying several properties of the toric Sasaki–Einstein geometries [30, 33]. On the other hand, we still have a $\mathbb{T}^3$ action and one may attempt to understand these geometries from a complex toric geometry viewpoint [111]. Below we will use the example of the $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ solution to illustrate some features of these geometries, that we expect to persist more generally.

A key property of toric Calabi-Yau singularities is that the image of the moment map associated to the $\mathbb{T}^3$ action is a convex polyhedral cone. The primitive normals to the facets of this cone can be projected to a plane, where they provide the toric diagram of the singularity. Equivalently, these normals correspond to the vanishing of different (Killing) vectors in $\mathbb{T}^3$, and thus define co-dimension two loci that are toric divisors in the Calabi–Yau cone, or equivalently calibrated three-manifolds in the Sasaki–Einstein base. These vectors may be extracted from an analysis of the explicit metric, and written in a basis for $\mathbb{T}^3$ they yield the toric diagram [144, 145]. Following these references, below we will employ this method for obtaining a toric diagram assocated to the $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ geometries, albeit one that will not be convex. As we will explain, this diagram is formally in 1–1 correspondence with that of the $Y^{p,q}$ geometries.

The analysis below will follow closely the discussion in [144,145] for the regularity of the five-dimensional $L^{a,b,c}$ toric Sasaki–Einstein metrics. This gives an alternative method to performing the regularity analysis of the metric, and in particular to determine the constraint $\mathfrak{p} < \mathfrak{q}$. The starting point is the local five-dimensional metric (C.3) depending on the parameter a. There are four codimension two fixed point sets, where the metric degenerates; these are at $x = x^+, x = x^-, \theta = 0$ and $\theta = \pi$. At each of these points a Killing vector has vanishing norm. We may introduce a 2π periodic coordinate for each of these angular directions at the degeneration loci by normalising the associated Killing vector such that its surface gravity, defined as

$$\kappa = \frac{\partial_\mu |V|^2 \partial^\mu |V|^2}{4|V|^2} \,, \tag{C.32}$$

is unity on the degeneration surface. With this choice of periodicity the Killing vector degenerates smoothly onto the degeneration surface.

The most general Killing vector one can construct is

$$V = S\partial_\alpha + T\partial_\phi + W\partial_\chi \,, \tag{C.33}$$

where S, T, W are three constants. This has norm

$$|V|^2 = \frac{w(x)}{4}(S + g(x)(T + \cos\theta W))^2 + \frac{a}{4}\left(\frac{U(x)}{w(x)}(T + \cos\theta W)^2 + \sin^2\theta W^2\right) . \tag{C.34}$$

The norm is a sum of three positive terms and therefore for it to vanish each of these terms must independently be zero. We find that the Killing vectors after being suitably normalised are

$$\begin{aligned} k^+ &= \frac{1}{x^+}\left(\partial_\alpha + x^+\partial_\phi\right) \ , \qquad & k^- &= \frac{1}{x^-}\left(\partial_\alpha + x^-\partial_\phi\right) \ , \\ k^0 &= \partial_\phi - \partial_\chi \ , & k^\pi &= \partial_\phi + \partial_\chi \ , \end{aligned} \tag{C.35}$$

where the superscript denotes the associated degeneration point. Clearly these four Killing vectors are not linearly independent as they span a three-dimensional space and therefore they must satisfy

$$Hk^+ + Jk^- + Kk^0 + Lk^\pi = 0 \ , \tag{C.36}$$

for some constant coefficients. As explained in [144] the constant coefficients must be integers. This follows because each of the Killing vectors generate 2π periodic translations, and therefore the coefficients must be rational. Then by taking integer combinations of translations around these circles one generates a translation which would identify arbitrarily close points. To prevent this from occurring one must take the coefficients to be integers which may be assumed to be coprime. One finds that the integers satisfy

$$H + J + K + L = 0 \ , \quad K = L \ , \tag{C.37}$$

and

$$\frac{H}{x^+} + \frac{J}{x^-} = 0 \ \Rightarrow \quad \sqrt{a} = \frac{H-J}{H+J} \ . \tag{C.38}$$

Taking into account the constraints above, we may redefine the integers H and J as

$$H - J = 2\mathfrak{q} \ , \quad H + J = 2\mathfrak{p} \tag{C.39}$$

for consistency with the previous section's notation, i.e. (C.29). Then from the constraint that $a > 1$ it follows again that $\mathfrak{p} < \mathfrak{q}$. Moreover, rewriting the linear relation between the vectors in terms of these two integers we find

$$(\mathfrak{p}+\mathfrak{q})k^+ + (\mathfrak{p}-\mathfrak{q})k^- - \mathfrak{p}k^0 - \mathfrak{p}k^\pi = 0 \ . \tag{C.40}$$

From this we can read off what in the GLSM language is called the "charge matrix" (up to an overall sign) to be

$$(\mathfrak{p}, \mathfrak{p}, -\mathfrak{p}+\mathfrak{q}, -\mathfrak{p}-\mathfrak{q}) \ . \tag{C.41}$$

Notice that this is formally identical to the charge matrix of $Y^{p,q}$ singularities, in particular the sum of all these charges vanishes. However, due to the different sign of $\mathfrak{p}-\mathfrak{q}$, here there are three positive charges and one negative for $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$, in contrast to the two positive and two negative for $Y^{p,q}$. In the Calabi–Yau context, these charges can be used to reconstruct the singularity (and all its resolutions) from the Kähler quotient $\mathbb{C}^4/\!\!/U(1)$. Then two charges of the same sign give rise to toric non-orbifold singularities, whereas three charges with the same sign produce a $\mathbb{C}^3/\mathbb{Z}_n$ orbifold. However, the cone over $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$ is *not* an orbifold singularity, as it follows from the preceding analysis, this is not in contradiction because the cone is not Kähler.

To extract a toric diagram from the previous analysis[8] we need to write the four vectors above in an effectively acting basis of $\mathbb{T}^3$. Locally the $\mathbb{T}^3$ action is generated by the vector fields ∂_α, ∂_ϕ and ∂_χ, but they do not give an effectively acting basis. Let an effectively acting basis of these Killing vectors be the set $\{e_1, e_2, e_3\}$, which are linear combinations of $\partial_\alpha, \partial_\phi, \partial_\chi$ and are taken to be suitably normalised such that all have period 2π. Any $SL(3,\mathbb{Z})$ transformation of this basis will also generate the effective $\mathbb{T}^3$ action. Writing the degenerating Killing vectors as a linear combination of the e_i and applying $SL(3,\mathbb{Z})$ transformations to bring the first row and column to a canonical form, this becomes

$$\begin{pmatrix} k^+ \\ k^- \\ k^0 \\ k^\pi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & A & B \\ 1 & C & D \\ 1 & E & F \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} . \tag{C.42}$$

Consider the degeneration surface defined by $x = x^+$ with degenerating Killing vector $k^+ = e_1$. The $\mathbb{T}^3$ fibration reduces smoothly to a $\mathbb{T}^2$ fibration over this surface which is spanned by $\{e_2, e_3\}$. At the intersection of this degeneration surface with the degeneration surfaces located at $\theta = 0$ and $\theta = \pi$ we have an additional degenerating Killing vector. Recall from previous arguments that this vector must be 2π periodic for the degeneration to be smooth. At $\theta = 0$ we have the Killing vector $Ce_2 + De_3$ degenerating on the surface. For this to be 2π periodic it is necessary that C and D are relatively prime, $\gcd(C, D) = 1$. A similar argument follows for the degeneration surface at $\theta = \pi$ and so we also have $\gcd(E, F) = 1$. Notice that there is no condition on A, B as the degeneration surface at $x = x^-$ does not intersect with the one at $x = x^+$. As $\gcd(C, D) = 1$ there exist integer solutions to $RC + SD = 1$ and therefore by an $SL(2,\mathbb{Z}) \subset SL(3,\mathbb{Z})$ transformation we may

[8]For a similar analysis in $L^{a,b,c}$ and conventions see [29, 146].

set $C=1, D=0$. We find

$$\begin{pmatrix} k^+ \\ k^- \\ k^0 \\ k^\pi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & A & B \\ 1 & 1 & 0 \\ 1 & E & F \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} . \tag{C.43}$$

Next, using the linear relation between the four Killing vectors we find

$$(\mathfrak{p}-\mathfrak{q})B - \mathfrak{p}F = 0 \ , \qquad (\mathfrak{p}-\mathfrak{q})A - \mathfrak{p} - \mathfrak{p}E = 0 \ . \tag{C.44}$$

These can be solved by

$$B = \mathfrak{p} \, , \qquad G = \mathfrak{p}-\mathfrak{q} \, , \qquad A = 0 \, , \qquad E = -1 \, , \tag{C.45}$$

and we obtain

$$\begin{pmatrix} k^+ \\ k^- \\ k^0 \\ k^\pi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & \mathfrak{p} \\ 1 & 1 & 0 \\ 1 & -1 & \mathfrak{p}-\mathfrak{q} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} . \tag{C.46}$$

We may now introduce three 2π periodic coordinates ψ_i for each of the three e_i, the change of coordinates from the original set is

$$\alpha = \frac{1}{x^+}(\psi_1 - \psi_2) + \left(\frac{\mu-\nu}{x^+} - \frac{\nu}{x^-} \right) \psi_3 \ , \quad \phi = \psi_1 \ , \quad \chi = -\psi_2 + \mu\psi_3 \ , \tag{C.47}$$

where the integers μ and ν satisfy $\mu(\mathfrak{p}-\mathfrak{q}) - \mathfrak{p}\nu = 1$ and are guaranteed to exist by the fact $\gcd(\mathfrak{p}, \mathfrak{p}-\mathfrak{q}) = 1$. With these coordinates the $\mathbb{T}^3$ action acts effectively.

Finally, we may read off the toric data from the matrix Λ: the four vertices are given by the rows of Λ, namely

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \begin{pmatrix} 1 \\ 0 \\ \mathfrak{p} \end{pmatrix} , \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} , \quad \begin{pmatrix} 1 \\ -1 \\ \mathfrak{p}-\mathfrak{q} \end{pmatrix} . \tag{C.48}$$

By an additional $SL(3,\mathbb{Z})$ transformation the vectors take the form

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \begin{pmatrix} 1 \\ \mathfrak{p} \\ \mathfrak{p} \end{pmatrix} , \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} , \quad \begin{pmatrix} 1 \\ \mathfrak{p}-\mathfrak{q}-1 \\ \mathfrak{p}-\mathfrak{q} \end{pmatrix} , \tag{C.49}$$

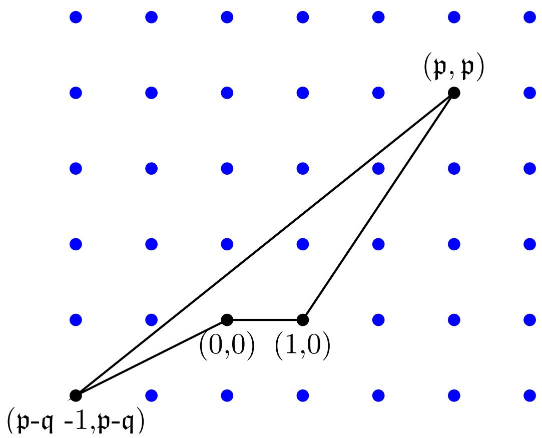

Figure C.1: Toric diagram for $\mathfrak{Y}^{\mathfrak{p},\mathfrak{q}}$. Notice that this is not convex. The figure represents the choice $\mathfrak{p} = 3$, $\mathfrak{q} = 4$.

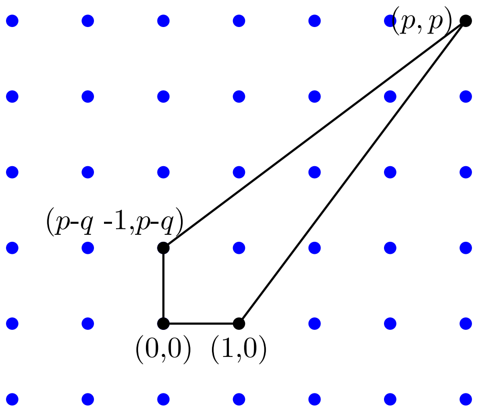

Figure C.2: Toric diagram for $Y^{p,q}$. The figure depicts the case $p = 4$ and $q = 3$.

which agree formally with the ones for the $Y^{p,q}$ Calabi–Yau singularity [28]. Notice however that because $\mathfrak{q} > \mathfrak{p}$, this no longer defines a convex polytope. For comparison, we contrast two examples of toric diagrams in the two cases in the figures C.1 and C.2.

Appendix D

AdS$_5$, $F_5 = 0$ appendix

D.1 Bilinear definitions and the orthonormal frame

In the appendix we define all the bilinears appearing in section 5. The scalar bilinears are

$$\begin{aligned} A &\equiv \frac{1}{2}(\bar{\xi}_1\xi_1 + \bar{\xi}_2\xi_2) \ , \\ A\sin\zeta &\equiv \frac{1}{2}(\bar{\xi}_1\xi_1 - \bar{\xi}_2\xi_2) \ , \\ S &\equiv \bar{\xi}_2^c\xi_1 \ , \\ Z &\equiv \bar{\xi}_2\xi_1 \ . \end{aligned} \tag{D.1}$$

The vector bilinears are

$$\begin{aligned} K^m &\equiv \bar{\xi}_1^c\gamma^m\xi_2 \ , \\ K_3^m &\equiv \bar{\xi}_2\gamma^m\xi_1 \ , \\ K_4^m &\equiv \frac{1}{2}(\bar{\xi}_1\gamma^m\xi_1 - \bar{\xi}_2\gamma^m\xi_2) \ , \\ K_5^m &\equiv \frac{1}{2}(\bar{\xi}_1\gamma^m\xi_1 + \bar{\xi}_2\gamma^m\xi_2) \ . \end{aligned} \tag{D.2}$$

The two-form bilinears are

$$\begin{aligned} W_{mn} &\equiv -\bar{\xi}_2\gamma_{mn}\xi_1 \ , \\ V_{mn} &\equiv -\frac{\mathrm{i}}{2}(\bar{\xi}_1\gamma_{mn}\xi_1 - \bar{\xi}_2\gamma_{mn}\xi_2) \ , \\ U_{mn} &\equiv -\frac{\mathrm{i}}{2}(\bar{\xi}_1\gamma_{mn}\xi_1 + \bar{\xi}_2\gamma_{mn}\xi_2) \ , \\ X_{mn} &\equiv \bar{\xi}_1^c\gamma_{mn}\xi_1 \ , \\ Y_{mn} &\equiv \bar{\xi}_2^c\gamma_{mn}\xi_2 \ , \end{aligned} \tag{D.3}$$

One finds that they satisfy the following algebraic relations

$$
\begin{aligned}
K_5 &= \sin\zeta\; K_4 + \mathrm{Re}\,[Z^* K_3] - \mathrm{Re}\,[S^* K]\ , && \text{(D.4)}\\
0 &= \sin\zeta V - U - \frac{\mathrm{i}}{2} K^* \wedge K + \mathrm{Re}\,[\mathrm{i} Z^* W]\ , && \text{(D.5)}\\
S^* X &= (1+\sin\zeta) W - (K_4 + K_5)\wedge K_3\ , && \text{(D.6)}\\
S^* Y &= (1-\sin\zeta) W^* - (K_4 - K_5)\wedge K_3^*\ . && \text{(D.7)}
\end{aligned}
$$

These relations may be computed by making use of Fierz identities, however we find it simpler to compute these by using an orthonormal frame which we shall construct below. Following [7] we take the basis of gamma matrices of Cliff(5) to be

$$
\begin{aligned}
\gamma^1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I \\
\gamma^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I \\
\gamma^a &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \tau^a
\end{aligned}
\qquad \text{(D.8)}
$$

where $\tau^a = -\mathrm{i}\sigma^a$ and σ^a are the Pauli matrices. In this basis the charge conjugation intertwiner is given by $C = I \otimes \tau^2$. we label the corresponding basis by e^i. We decompose the spinors ξ_i as $s_i \otimes \theta_i$ where s_i are spinors of Cliff(3) and θ_i spinors of Cliff(2). At the moment the basis is completely arbitrary which allows us to impose that the two vectors K_4 and K_5 lie in the $(e^1 - e^2)$ plane and in particular K_5 to be parallel with e^1. We find

$$
s_1 = \sqrt{2}\begin{pmatrix} \cos\theta\cos\phi \\ -\sin\theta\sin\phi \end{pmatrix}\ , \quad s_2 = \sqrt{2}\begin{pmatrix} \sin\theta\cos\phi \\ \cos\theta\sin\phi \end{pmatrix} \qquad \text{(D.9)}
$$

where we have set $\bar{\theta}_i\theta_i = 1$ and added suitable normalization to enforce $A = 1$. We can now write the scalar and vector bilinears as functions of θ, ϕ, θ_i. Requiring $\sin\zeta = 0$ implies that $\cos 2\theta = 0$ otherwise $\cos 2\phi = 0$ which then implies $K_5 = 0$. Choosing K_3 to lie in the (e^3-e^4) plane one can choose:

$$
\theta_1 = \begin{pmatrix} e^{\mathrm{i}\alpha} \\ 0 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 0 \\ e^{\mathrm{i}\alpha} \end{pmatrix} \qquad \text{(D.10)}
$$

from which we obtain the final form of the vector bilinears

$$K_5 = \cos 2\phi e^1 \ , \quad K_4 = -\sin 2\phi e^2 \ , \quad K_3 = \sin 2\phi (e^4 - \mathrm{i} e^3) \ ,$$
$$K = \mathrm{e}^{2\mathrm{i}\alpha} e^1 - \mathrm{i} \sin 2\phi e^{2\mathrm{i}\alpha} e^5 \ , \tag{D.11}$$

and the one non-trivial scalar bilinear

$$S = -\mathrm{e}^{2\mathrm{i}\alpha} \cos 2\phi \ . \tag{D.12}$$

The two-forms in terms of this orthonormal basis are

$$U = -\sin 2\phi e^{15} \ , \quad V = e^{34} - \cos 2\phi e^{25} \ , \quad W = (\mathrm{i} \cos 2\phi e^5 - e^2) \wedge (e^4 - \mathrm{i} e^3) \ ,$$
$$X = \mathrm{e}^{2\mathrm{i}\alpha} (\sin 2\phi e^1 + \cos 2\phi e^2 - \mathrm{i} e^5) \wedge (e^4 - \mathrm{i} e^3) \ ,$$
$$Y = \mathrm{e}^{2\mathrm{i}\alpha} (-\sin 2\phi e^1 + \cos 2\phi e^2 + \mathrm{i} e^5) \wedge (e^4 + \mathrm{i} e^3) \ . \tag{D.13}$$

D.2 More details on the solution of [1]

In this appendix we present details about the derivation of (5.74)-(5.79). We make no claims that all the work in this appendix is original, only the final expressions (5.74)-(5.79). As pointed out in the text we were unable to solve the equations of the classification in order to recover this solution, in hindsight this was to be expected as it solves very non-trivial equations compared to the ansatz we have considered. Instead we found the Killing spinor of the NATD-T solution and from it constructed the spinor bilinears which allowed us to recover the solution. One may solve the Killing spinor equations directly for the NATD-T solution however this is very difficult and may be avoided. Instead one can use the Killing spinors of $T^{(1,1)}$, which are relatively simple to find, and transform them under the corresponding NATD and T dualities. It is this method that we present below.

The Buscher rules [147] give the transformation of the NS-NS sector under T-duality whilst [148] first gave the transformation of the RR-fluxes. The transformation of the Killing spinors was found in [149]. It is also well known how the geometry changes under NATD, see [150] for the transformation of the NS-NS sector, though we shall follow the conventions in [151]. The transformation of the RR-fluxes was found in [152] whilst in [153] it was found how a Killing spinor transforms under NATD. We shall briefly present the transformation of the Killing spinors under both NATD and T-duality for the ease of the reader.

Under a NATD or T-duality there is some ambiguity with the transformation of the vielbeins. Left and right movers of the world-sheet have different transformation properties and therefore define two different frame fields. These two frames must be equivalent as they define the same geometry and so are related by a Lorentz transformation of the form:

$$\hat{e}_+ = \Lambda \hat{e}_- \ . \tag{D.14}$$

This Lorentz transformation induces an action on spinors by the matrix Ω which satisfies

$$\Omega^{-1}\Gamma^a\Omega = \Lambda^a{}_b\gamma^b \ . \tag{D.15}$$

Type IIB supersymmetry is parametrised by two $d = 10$ Majorana-Weyl spinors of the same chirality whilst type IIA is paramtrised by two $d = 10$ Majorana-Weyl spinors of opposite chirality. We shall denote these two spinors generically as χ_1 and χ_2, their chiralities are unimportant for the calculation and so we do not distinguish their chiralities. Under a NATD or T-duality

$$\chi_1 \to \chi_1 \qquad \chi_2 \to \Omega^{-1}\chi_2 \ . \tag{D.16}$$

where for a T-duality along a Killing vector, ∂_x, Ω takes the form

$$\Omega^{-1}_{U(1)} = -\frac{1}{\sqrt{G_{xx}}}\Gamma_{11}\Gamma_x \ , \tag{D.17}$$

where x is a curved index on Γ_x. Under a NATD, with respect to an $SU(2)$ isometry along the flat directions $1, 2$ and 3, Ω takes the form

$$\Omega^{-1}_{SU(2)} = -\frac{\Gamma^{(11)}}{\sqrt{1+\zeta^2}}(\Gamma^{123} + \zeta_a\Gamma^a) \ , \tag{D.18}$$

where for our purposes

$$\zeta^1 = \frac{x_1\cos\xi}{L^2\lambda_1\lambda} \ , \quad \zeta^2 = \frac{x_1\sin\xi}{L^2\lambda_1\lambda} \ , \quad \zeta^3 = \frac{x_2}{L^2\lambda_1^2} \ . \tag{D.19}$$

Note that both Ω's defined above are unitary in our basis.

To begin we solve the Killing spinor equations of the Klebanov-Witten solution, $T^{(1,1)}$, in the canonical vielbein basis for performing the NATD

$$\begin{aligned} e^{\theta_1} &= L\lambda_1 \mathrm{d}\theta_1 \ , \quad e^{\phi_1} = L\lambda_1\sin\theta_1\mathrm{d}\phi_1 \ , \\ e^{1,2} &= L\lambda_1\tau_{1,2} \ , \quad e^3 = L\lambda(\tau_3 + \cos\theta_1\mathrm{d}\phi_1) \ , \end{aligned} \tag{D.20}$$

where τ_i are the left invaraint $SU(2)$ one-forms. With this basis, the Killing spinors are

$$\chi_1 = \frac{1}{2}\begin{pmatrix}1\\0\\\mathrm{i}\\0\end{pmatrix} , \qquad \chi_2 = \frac{1}{2}\begin{pmatrix}\mathrm{i}\\0\\-1\\0\end{pmatrix} , \tag{D.21}$$

where the choice of normalization is for later convenience. From these two spinors we may construct ξ_1 and ξ_2 as used in the classification

$$\xi_1 = \chi_1 + \mathrm{i}\chi_2 \ , \quad \xi_2 = \chi_1 - \mathrm{i}\chi_2 \ , \tag{D.22}$$

note that it is the χ's that transform as (D.16) and not the ξ's. Under the NATD the Killing spinors become

$$\chi_1 \to \chi_1 \ , \quad \chi_2 \to \Omega^{-1}_{SU(2)}\chi_2 \ , \tag{D.23}$$

whilst the vielbeins that change are[1]

$$\begin{aligned}
\hat{e}^1 &= -\frac{\lambda_1}{LQ}\big[((L^4\lambda_1^2\lambda^2 + x_1^2)\cos\xi + L^2\lambda^2 x_2\sin\xi)\mathrm{d}x_1 \\
&\quad + x_1(x_2\cos\xi - L^2\lambda_1^2\sin\xi)(\mathrm{d}x_2 + L^2\lambda^2(\mathrm{d}\xi + \cos\theta_1\mathrm{d}\phi_1))\big] \\
\hat{e}^2 &= -\frac{\lambda_1}{LQ}\big[((L^4\lambda^2\lambda_1^2 + x_1^2)\sin\xi - L^2\lambda^2 x_2\cos\xi)\mathrm{d}x_1 \\
&\quad + x_1(L^2\lambda_1^2\cos\xi + x_2\sin\xi)(\mathrm{d}x_2 + L^2\lambda^2(\mathrm{d}\xi + \cos\theta_1\mathrm{d}\phi_1))\big] \\
\hat{e}^3 &= -\frac{\lambda}{LQ}[x_1x_2\mathrm{d}x_1 + (L^4\lambda_1^4 + x_2^2)\mathrm{d}x_2 - L^2\lambda_1^2x_1^2(\mathrm{d}\xi + \cos\theta_1\mathrm{d}\phi_1)].
\end{aligned} \tag{D.24}$$

One now has all the information to perform the T-duality. After both dualities the $T^{(1,1)}$ spinors become

$$\chi_1 \xrightarrow[\text{NATD}]{} \chi_1 \xrightarrow[\text{T}]{} \chi_1 \ , \quad \chi_2 \xrightarrow[\text{NATD}]{} \Omega^{-1}_{SU(2)}\chi_2 \xrightarrow[\text{T}]{} \Omega^{-1}_{U(1)}\Omega^{-1}_{SU(2)}\chi_2 \ . \tag{D.25}$$

One may now compute all the spinor bilinears. One finds for the scalar bilinears

$$A = 1 \ , \tag{D.26}$$

$$\sin\zeta = 0 \ , \tag{D.27}$$

$$Z = 0 \ , \tag{D.28}$$

$$S = -\frac{\lambda_1^2 x_1 \sin\theta e^{\mathrm{i}\xi}}{\sqrt{W}} \ . \tag{D.29}$$

[1]Notice that we have rotated $\hat{e}^1$ and $\hat{e}^2$ with respect to those presented in appendix 6 of [1]. We have also added some extra factors of λ and λ_1 which we found to be missing.

From S one finds

$$\xi = -\psi \ , \qquad \eta\mu = \frac{\lambda_1^2 x_1 \sin\theta}{\sqrt{W}} \ . \tag{D.30}$$

Moreover one sees that the warp factor arises from putting the $d = 10$ metric into Einstein frame and therefore we have the identification $\mathrm{e}^{2\Delta} = \mu^{-1/2} = \mathrm{e}^{-\Phi/2}$. From this we find

$$\eta = L^2 \lambda_1^2 x_1 \sin\theta_1 \ . \tag{D.31}$$

One is able to find the one-form bilinears K_5 and K from this information by using (5.29) and (5.30) and we may use this as a check for the result defined directly from the Killing spinors. Computing the one-form bilinears form the Killing spinors one finds

$$\begin{aligned}
K &= \frac{L\lambda\lambda_1^2 e^{i\xi}}{\sqrt{W}}(i(\sin\theta \mathrm{d}x_1 + x_1\cos\theta \mathrm{d}\theta) - x_1 \sin\theta \mathrm{d}\xi) \ , & \text{(D.32)}\\
K_5 &= -\frac{L\lambda\lambda_1^4 x_1^2 \sin^2\theta}{W}\mathrm{d}\xi \ , & \text{(D.33)}\\
K_4 &= \frac{\lambda(-x_1\cos\theta_1 \mathrm{d}x_1 - x_2\cos\theta_1 \mathrm{d}x_2 + L^4\lambda_1^4 \sin\theta_1 \mathrm{d}\theta_1 + x_2 \mathrm{d}\phi_1)}{L\sqrt{W}} \ , & \text{(D.34)}\\
K_3 &= \frac{\lambda\lambda_1^2}{LW}[x_1x_2\sin^2\theta_1 \mathrm{d}x_1 + (L^4\lambda_1^4 + x_2^2)\sin^2\theta_1 \mathrm{d}x_2 + x_1^2\cos\theta_1 \mathrm{d}\phi_1 & \\
&\quad + \mathrm{i}L^2\sqrt{W}(\cos\theta_1 \mathrm{d}x_2 + x_2\sin\theta_1 \mathrm{d}\theta_1 - \mathrm{d}\phi_1)]. & \text{(D.35)}
\end{aligned}$$

Finally, using the redefinitions used in the classification (5.30) and (5.31), one recovers (5.75)-(5.79). The change of coordinates (5.83)-(5.86) follows from noticing that ϕ can be identified with x and then observing that certain combinations of $\mathrm{d}x_i$ and $\mathrm{d}\theta_1$ appear only. From these combinations by adding suitable functions and requiring that they are closed one recovers the change of coordinates presented.

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